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Non-monotonic Logic via Topology

H.H. Jurjus

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Non-monotonic Logic via Topology

Proefschrift

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Preface

This thesis is the result of research that started with two observations and three general aims. In the first place, it was observed that most non-monotonic formalisms in the literature depict non-monotonic reasoning as a complicated mechanism, while on the other hand, human non-monotonic reasoning seems a trivial procedure. In the second place, it was realized that (human) non-monotonic reasoning is not necessarily less rational than, say, mathematical reasoning. These seemingly unconnected observations gave rise to the somewhat unusual approach to non-monotonic logic of this thesis.
While non-monotonic reasoning is typically considered and studied as a subdiscipline of artificial reasoning, we aimed at an approach that depicts non-monotony as a respectable subject of (mathematical) *logic*. A first important task, for example, is to extend classical propositional logic with "defeasible" implication in a convincing way. Defeasible implication should be taken seriously as a sensible alternative for "ordinary" implication.
A second general aim was to depict non-monotonic reasoning as a *natural* phenomenon and to keep our formalisms as simple as possible. The third major aim was to use non-monotonic reasoning for reflections on *foundational issues*. For example, it seems that human reasoning is "too non-monotonic" to be mathematically formalizable. Is it possible to prove or to understand this "non-formalizability"?
More concretely, we presume that there exists a "calculus of rules-with-possible-exceptions," governed by general principles that are themselves rules-with-possible-exceptions. We will be interested in non-monotonic formalisms in which at least some of the rules of inference are defeasible. Do such formalisms exist? How to construct them? How do they behave in comparison to other non-monotonic formalisms?
Before we can address such issues, however, a lot of work has to be done. We will use elementary topology to define a semantics of defeasible implication. The definition will generalize situations like "if ℓ is a line and P is a point in the Euclidean plane, then typically P is not on ℓ ; but if P is known to be on ℓ, this conclusion is no longer sanctioned." Starting from these topological notions, we will construct simple formalisms in which, for example, the rule of monotony is valid-up-to-possible-exceptions. This will have interesting foundational consequences.

We will not aim at constructing exact descriptions of practical reasoning, nor at prescriptions for systems of artificial reasoning. Nevertheless, it is hoped that this thesis will contribute to the understanding of non-monotony as it occurs in practical reasoning and as it should occur in artificial reasoning.

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Some standard notations, as used in this thesis

$\{... \mid ...\}$	the set of all ... such that ...
$... \in ...$	... is an element of the set ...
$... \ni ...$	the set ... contains the element ...
$... \subseteq ...$	... is a subset of ...
$... = ...$	... equals ...
$... \neq ...$	... is distinct from ...
$\varnothing$	empty set; (unique) set without elements
$\mathcal{P}(X)$	power set of the set X; the set of all subsets of X
$A \times B$	the set of all pairs (a, b) such that $a \in A$ and $b \in B$
$A \cap B$	the intersection of A with B; the set of all elements that A has in common with B
$A \cup B$	the union of A and B; the set of all elements that are in A, or in B, or in both
$\bigcup_{...} A$	the union of all sets A such that ... ; $\{ x \mid x \in A$ for some set A satisfying ... $\}$
$X \setminus A$	the set of all elements of X that are not in A
A^c	(relative) complement of A; $X \setminus A$ for some set X
$... \wedge ...$	... and ...
$... \vee ...$	... or ...
$\neg ...$	not ...
$\Rightarrow$	implies
$\Leftrightarrow$	(only) if; if and only if
$\forall...[...]$	for all ... : ...
$\mathbb{N}$	the set of all natural numbers; {0, 1, 2, ...}
$\mathbb{Z}$	the set of all integers; {..., -2, -1, 0, 1, 2, ... }
$\mathbb{Q}$	the set of all rational numbers (fractions, including negative numbers)
$\mathbb{R}$	the set of all real numbers (the set of points on a line)
$\mathbb{C}$	the set of all complex numbers (the set of points in a plane)
$\rightarrow$	see 5.7 and p. 43
$\vdash$	see p. 54
$\vdash_{\leq}$	see 7.3
$\square$	end of proof

Contents

Chapter I
Introduction

Although the core of this thesis consists of mathematical definitions that could very well speak for themselves, this chapter will provide a number of possible motivations for them. The presentation will be rather informal, and many terms, such as "reasoning," "logically correct" or "classical logic," will be used without further specification.

§1 Roots

The subject of this dissertation is non-monotonic logic. This term refers to the investigation of formal systems of logic that fail to satisfy the principle of monotony: "if some collection of premisses supports some conclusion, then any larger collection of premisses will support that conclusion as well."

The interest in such systems originally arose from research in artificial intelligence, in particular in connection with what is usually called the frame problem (to be explained below). Besides that, logicians have long been aware of the existence of non-deductive reasoning. But formalizing such types of reasoning leads to severe difficulties unless a non-monotonic formalism is accepted as a sensible option. Let us consider some examples.

– The *frame problem* is about finding an efficient representation of knowledge about changing domains and of the typical inertia principle that people seem to apply so easily in reasoning about them (for example, when reading a novel or hearing a fairytale). Intuitively, this inertia principle could be stated as "things keep on going as usual, unless we are told otherwise." For instance: on the table there is a green block, a blue block and a red block. The blue block is removed. What blocks are left on the table? The obviously intended answer is "(only) the green block and the red block." But, for example, the assumptions

```
At(table, block1, t0), green(block1),
At(table, block2, t0), blue(block2),
At(table, block3, t0), red(block3),
Removed(block2, t1)
```

(where `t0` and `t1` denote time-points)

do not imply, e.g.,

```
At(table, block3, t2)
```

nor `¬At(table, block4, t2)`.

And indeed, the "obviously intended answer" is not logically implied by the premisses, unless it is assumed that no other blocks were on the table than the ones mentioned and that no other blocks were removed than the one mentioned. Finding out, for example, that there was a purple block on the table before we started and that the red block was also removed will force us to withdraw the formerly correct answer. The problem is to automatize the process of finding the intended answer, thus formalizing, in some way, the inertia principle involved. As this is a process in which some knowledge leads to some conclusions, we may refer to this process as (a non-standard type of) *reasoning*. The idea is to view the answer as implied by the premisses, where implication has some non-standard interpretation. Some have suggested that the word entailment should be used rather than implication.

Formalizing this type of reasoning (i.e., completely describing the entailment relation) in its full generality is a delicate matter, as is apparent from the following example: Fred is in the kitchen, sitting at a white table, admiring his green wallpaper. One week later, we visit him again. We typically expect that the white table will still be there and that the wallpaper will still be green (as a matter of fact, we silently assume that the kitchen will still be there). But do we expect that Fred will still be in the kitchen? A sentence like `In(kitchen, table)` should, once true, typically remain true, while `In(kitchen, Fred)` should not. But `In(coffin, Fred)`, in its turn, should. A convincing formal treatment of a useful and sensible entailment relation is still pending.

Some other sources of non-monotonic reasoning :

– An interesting task for artificial intelligence in general is the design of an intelligent (artificial) secretary with the ability to understand ordinary text and conversation. Such a system should, among other things, be able to handle trivial-but-withdrawable conclusions. For example, if we order a cup of tea with sugar, the system should understand that we will not be happy with a cup of tea with both sugar and diesel oil in it. But this trivial conclusion should not be drawn if a cup of tea with both sugar and diesel oil is explicitly ordered.

– Formalizing *inductive reasoning* may also lead to non-monotony: after seeing 1273 swans, each one of them white, we draw the conclusion "all swans are white." Observing a black swan forces us to withdraw that conclusion. Inductive reasoning is related to the type of reasoning involved in tasks like: "complete the following row: 1, 3, 2, 4, 3, 5, ... ". The

obviously intended answer ("4, 6, 5, 7, ...") is withdrawn after hearing that the next number is not 4.
– *Abductive* (diagnostic) *reasoning* : both diseases D and D' cause symptom P. D causes symptom Q as well. A patient showing P but not Q is concluded to have disease D'. Hearing about another disease, D", causing P but not Q, would force us to withdraw that conclusion.
– *Dialectical reasoning* (reasoning by arguments and counterarguments) is a very common type of reasoning, used, e.g., in court, in political debate, and in meetings. In court, only available information can be taken into account, and a decision has to be made within a limited amount of time. The court's conclusions will be considered correct provided certain rules have not been violated (if it was a fair trial). Nevertheless, it is possible that new information will shed new light on a case, or that the availability of more information would lead to a different result.

The examples mentioned so far depict non-monotonic logic as investigating formalizations of non-standard types of reasoning; reasoning that is, in principle, incorrect. Indeed, in the literature on non-monotonic logic, non-monotonic formalisms are usually thought of as models of certain types of human reasoning that deviate rather drastically from correct reasoning, but that are remarkably efficient and useful for certain practical purposes. Non-monotonic formalisms, then, are acknowledged to be models of incorrect reasoning. Sometimes it is even argued (e.g., in [Flach 95]) that logic, investigating human reasoning, should not, in the first place, be concerned with distinguishing correct from incorrect reasoning, but rather with modelling or describing all kinds of human reasoning. (We do not agree with this position; see §3 for more remarks on the purpose of logic.) In both cases, it is held that we should be interested in logics differing from classical logic because humans happen to reason incorrectly most of the time.
One of the primary motivations for this dissertation is to take non-monotony more seriously. This can be done in several ways. One possibility is to point out some blind spots of classical logic, formalize the neglected phenomena, and find out that the resulting formalism is non-monotonic. After all, classical logic (say, first order predicate logic) does in fact neglect some phenomena typical for human reasoning that could hardly be called irrational or incorrect and some of these phenomena would lead to non-monotonic formalisms.
For example, humans usually know how to take their own fallibility into account. In reasoning, they might take into account possible failure of as-

sumptions, of arguments, or of necessary silent presumptions. Reasoning that cannot deal with its own failure might be considered irrational. However, classical logic traditionally avoids this issue altogether. Below, we will see how this phenomenon may lead to non-monotony. Another phenomenon neglected by classical logic that may lead to non-monotony is the multitude of possible interpretations that implication might have, each one distinct from the conception of implication as used in predicate logic (as well as in mathematical reasoning in general). That another definition of implication may lead to non-monotony can be seen, e.g., in conditional logic (see [Lewis 73]), or in the rest of this dissertation. Both phenomena will play a role in the things to follow.

Another possibility, not necessarily incompatible with the former, is to realize that a formal system, such as predicate logic, is best seen as a mathematical model (of something that we have called a type of reasoning). But a model is like a caricature, emphasizing certain aspects, neglecting others. And it is not uncommon for an object to have two different models. This leads us to the first main point of departure of this dissertation: *we want to view both classical logic and non-monotonic logic as caricatures of the same type of reasoning*. Consequently, we will restrict our attention to those forms of non-monotony that are compatible with this point of view. This excludes most examples in the beginning of this section.

The type of non-monotony that we want to study is best illustrated by the following example. *John is always home at 6 o'clock, it's 6 o'clock now, and John is not home (now).*

What conclusions can be drawn from this combination of assumptions?

Let us rephrase "John is always home at 6 o'clock" as "if it is 6 o'clock, then John is home." Then the assumptions may be written as $p \rightarrow q$, p, $\neg q$, where p denotes "it is 6 o'clock now" and q denotes "John is home now."

If we interpret implication ("$\rightarrow$") as implication-without-possible-exceptions, as is usual in mathematics, this set of assumptions is contradictory, and we may conclude anything from it, including "I am the emperor of China," as well as "John is home now." However, in everyday life, we would usually take into account the possible failure of "John is always home at 6 o'clock" in exceptional cases. Hence, we would not consider "John is home now" to be a sensible conclusion to those three assumptions. In particular, the combination of assumptions is not inconsistent.

On the other hand, knowing how to handle possible exceptions to an implicational statement does not necessarily mean that we do not take the statement seriously (as a statement expressing implication). For example, the combination of just the two assumptions "John is always home at 6 o'clock" and "it is 6 o'clock now" should allow "John is home now" as a sensible conclusion.

Symbolically represented, classical logic accepts the following argument :

$$\dfrac{\dfrac{\dfrac{p \to q,\quad p,\quad \neg q}{p \to q,\quad p}}{q} \qquad\qquad \neg q}{\bot}$$

However, with a less pedantic interpretation of implication *as well as inference*, we could accept $\dfrac{p \to q,\ p}{q}$ as valid, but not $\dfrac{p \to q,\ p,\ \neg q}{q}$ (which shows that the resulting inference relation would be non-monotonic), nor $\dfrac{p \to q,\ p,\ \neg q}{\bot}$ (since "$\to$" amounts to implication-with-possible-exceptions).

In Chapters 2 and 3, we will present mathematical definitions of implication-with-possible-exceptions and of an inference relation that behave as indicated.

If we say that implication as used in everyday reasoning rarely amounts to implication-without-possible-exceptions, it is important to understand that this is not a linguistic issue. Although it is also true that conditional sentences as used in natural language may allow exceptions, we consider implication to be an instrument used by humans to organize or represent knowledge *in their mind*, without the help of natural language. It just happens not to be an essential aspect of implication that it cannot have exceptions. On the contrary, implication signifies the existence of a rule. But rules may have exceptions and nevertheless be valid. Hence, in this dissertation, we will take the position that it is natural to interpret implication as implication-as-a-rule-with-possible-exceptions. We will defend this viewpoint, not by argumentation leading to a conclusion, but by investigating its mathematical consequences and possibilities.

We want to study non-monotonic logic as a subdiscipline of *logic*, using mathematical methods. There is reason to doubt, however, whether the mathematical methods typically used in logic will suffice. The activity, at present, in the field of non-monotonic reasoning can roughly be divided into two sectors. The first sector consists of proposals for concrete prescriptions, usually stemming from an attempt to simulate human reasoning processes. Most of them either use probabilistic considerations (e.g., [Adams 75], [Bacchus 90]; see also [Geffner 92]), try to model reasoning-by-lack-of-information (e.g., default logic and auto-epistemic logic, see [Brewka 91]), or involve some minimalization of possible models, minimizing, for example, the collection of unexplained exceptions to rules or unexpected changes in a changing domain (e.g., circumscription, see [Brewka 91]). Since most of these "concrete" approaches were designed with a number of concrete examples in mind, they typically lack generality. The general tendency in this sector is towards more and more complicated machinery (for example, [Nait Abdallah 95], 715 pages).

The second sector consists of more axiomatic approaches, in which it is asked what properties some reasonable system should or may have (e.g., [KLM 90], [Flach 95], but we can also consider [AGM 85] to be in this sector). These approaches mimic classical logic in that completeness results are used to establish connections between semantical constructions and axiom systems. Since we aim at a logical, non-adhoc approach, it will be clear that it is this sector that will be of interest to us. But the approaches in this sector have problems analogous to those in the first sector. Most proposed axiom systems seem either too weak or too strong. Which brings us to another main issue with which we began our inquiry: *why is it so difficult* to find a convincing formal treatment of these types of reasoning, while, at the same time, humans seem to handle them so easily?

Of course, there is a variety of essentially different types of non-monotonic reasoning and it is to be expected that, say, abductive reasoning will need an other formalization than, say, inductive reasoning. Moreover, a typical "common sense" argument may involve a combination of several different types of inference. There is no doubt that this drastically complicates the task of formalization. However, looking at the attempts so far, this does not seem to be the main difficulty, since most of the approaches restrict their attention to a single type of reasoning, proposing some axiom system and then finding some counterintuitive examples of the very type of reasoning they were intended to axiomatize. One possible answer to this question will be informally sketched in §2 below. At this stage, two comments should be made. In the first place, answering the question of why it is so difficult should not be confused with solving the difficulty itself. In the second place, since we will present no more than an informal sketch, it is not and cannot be claimed that §2 provides the only answer. Our answer will essentially be to indicate a lack of expressive power of the mathematical modelling methods used in most approaches, in particular, the way in which rules of inference are interpreted.

It is in this context that one may ask, *what would happen if we interpret rules of inference, being implicational statements, as rules-with-possible-exceptions?* This dissertation is, by and large, an attempt to provide mathematical methods by which this (latter) question may be investigated.

§2 On the Nature of Models

This section will informally present motivations for the contents of this dissertation, resulting from reflection on the mathematical methods that are (to be) used in logic. In particular, from reflection on the notion of a *model.*

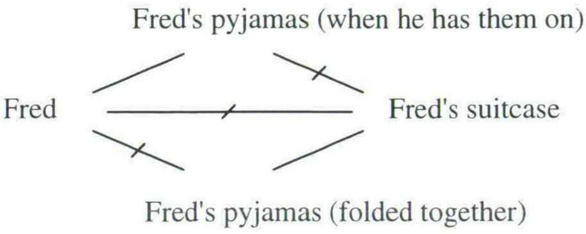
The word "model" is usually associated with resemblance, a model being either the original or an image of something else. In mathematics, however, any mathematical structure is called a model, and the term is not associated with resemblance. Some models are acknowledged to have been inspired by some product of human imagination, but, once a useful model has been completely specified, the original source of inspiration is typically ignored, nomenclature excepted. In this section (only), we will use the term "model" to denote any image (of something else), usually, but not necessarily, an abstract, schematic presentation. A mathematical model, then, is a mathematical structure *intended to bear some resemblance to something else.* A single object may have more than one decent model, analogous to the possibility of having several different pictures of a single object. In addition, a model is like a caricature, exaggerating certain properties, ignoring others. It typically resembles the original only in some respects. Finally, a model is like an approximation: susceptible to improvement or *precization.* Let us give an example.

Fred fits into his pyjamas. His pyjamas fit into his suitcase. Fred does not fit into his suitcase.

A possible model of the contents of these sentences is:

Fred — Fred's pyjamas — Fred's suitcase

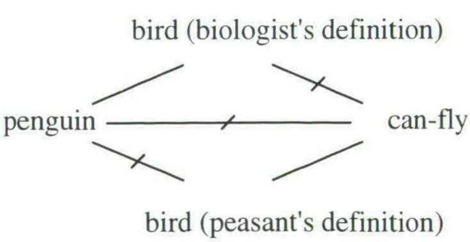
In this model, the relation "...fits into..." is represented by an intransitive relation (that is, a relation ...R... such that aRb, bRc does not imply aRc). This does not necessarily imply, however, that the relation we originally had in mind really *is* intransitive, as is apparent from the following precization of this model :



The former model can still be considered to be a decent and accurate model, however.

From this example, several things can be learned.
The first is that, under certain circumstances, a transitive relation may be adequately represented by an intransitive relation. Equivalently, claiming that some intransitive relation is an accurate representation of a certain relation does not amount to claiming that the relation at issue is really intransitive. To return to the subject of non-monotony, let us rephrase the example.
If something is a penguin, it is a bird. If something is a bird it (has the property that it) can fly. If it is a penguin, it cannot fly.
The corresponding model would be :

and the precization:



In this way, a formal system of logic in which the relation of implication is intransitive could be an accurate mathematical model of a type of reasoning in which implication is a transitive relation. Likewise, it should be possible to design a non-monotonic formal logic that can be seen as a decent and accurate model of an ordinary, monotonic type of reasoning. In this thesis, however, we will not proceed in this direction.

Another point is that something like Fred's pyjamas, which seems to be representable as a single object without any danger of confusion, suddenly seems an ambiguous concept in the more precise model. This phenomenon is typical for the kind of notions that arise in practical argumentation. Such notions typically are abstractions, enabling us to classify different objects or perceptions as equivalent-in-some-respect. The ability to form and handle abstract notions of this kind is a fundamental and indispensable instrument for humans to organize real-world knowledge. Even the perception of a single object as a single object is essentially due to an identification of various perceptions.
However, as the example shows, two notions equivalent in some respects may fail to be equivalent in some other respects. Hence, any mathematical structure, being a fixed set of elements with a *fixed* equivalence relation (namely "equality"), when intended to model a number of notions stemming from practical argumentation, will typically be susceptible to precization, like the model in our example. But this precization, in its turn, will also be susceptible to precization, and so on.

A related issue is the following. It is common knowledge among mathematicians that natural language does not lend itself to precise definitions or descriptions of any kind. Any natural language description is susceptible to misunderstanding. Attempts to clarify a particular description by providing an additional explanation in natural language leaves just one more description, susceptible of (other) misunderstanding. Mathematicians use artificial notations and notions, a "language" felt to be superior to natural languages, in that respect. However, for the delineation of notions that arise in practical argumentation, there is no reason to suppose that this mathematical language is any better. But it would be downright strange if notions (seemingly) unfit for mathematical treatment were to be discarded as unsensible on these grounds rather than blaming a deficiency in the (mathematical) language used. Consider the following metaphor:

> Someone is looking at a piece of paper and claims he has a precisely determined point in mind. A sceptical opponent demands that he draws it. The person draws some spot with a pencil, whereupon his opponent gets a magnifying glass and denies the claim. Using the same magnifying glass and more accurate drawing material, a more precise spot is located. The opponent gets a stronger magnifying glass and, again, denies the claim. After some attempts, the person refuses further discussion, blames his drawing material but maintains his original claim. The opponent's denial remains ...

Mathematicians consider a spot drawn with a pencil to be no more than an approximation of a real point, while a real point is "infinitely precise," hence, essentially undrawable. In view of the above metaphor, it is good to know that there is another treatment of the same geometrical intuitions, namely Johannes Hjelmslev's approach called "natural geometry" (see [Hjelmslev 23]), which takes "rough spots" seriously, and ignores the infinitely precise points as unnecessary idealizations. We could compare notions as arising from practical reasoning to precise points, and we already depicted their representations in mathematical models as approximations. Thus, it is to be expected that Hjelmslev's approach is of some value for the study of mathematical representation of notions arising from practical reasoning. Unfortunately, the mathematical elaborations of Hjelmslev's geometry, as developed so far, although geometrically very convincing, can not directly be used for our purposes. In this thesis, instead of adapting Hjelmslev's geometry for our purposes, we will use elementary topological notions, which is intended to have the same effect. However, from a mathematical viewpoint, the connection between the upcoming definitions and Hjelmslev's geometrical ideas will not be apparent.

The next point to be learned from the example about Fred's pyjamas is that, apparently, many situations that are inconsistent according to classical logic, do have a plausible interpretation if we take the above-mentioned phenomenon into account. For example, "R is a transitive relation," "aRb," "bRc," "not: aRc" is, classically, an inconsistent combination of assumptions, but it is not entirely unthinkable. It is not too difficult to define a class of generalized models and a corresponding definition of truth that captures this phenomenon. However, such a class of models, when properly defined, will contain plausible representations of many more situations that are inconsistent according to classical logic. It is to be expected, then, that within such a class of models, only very few "laws of logic" will be valid. Hence, practical reasoning is probably *not susceptible to precise description* using, for example, certain axiom system. To gain insight, nevertheless, into practical argumentation, other tools are needed.

For example, perhaps rules exist that are valid-(only)-up-to-possible-exceptions and that capture essential features of practical reasoning. The elaboration of this idea requires some (mathematical) definition of what it means for a rule to be valid-up-to-possible-exceptions. This is exactly what we aim at. The following might serve to elucidate our intentions.

In general, the study of complex systems of any kind requires adequate instruments. A well-known example is the Euclidean axiom system. Before the emergence of these axioms, geometry was already a well developed field of knowledge, with numerous theorems. Euclid managed to organize this large amount of knowledge in a useful way, and to distinguish between (geometrically) relevant and irrelevant issues (such as, "what exactly *is* a point or a straight line?"), solving many controversies by establishing efficient conventions and habits. One of these was the habit of interpreting implicational statements as implication-without-exceptions. Thus, the notion of implication-without-exceptions proved to be an adequate instrument for organizing that particular amount of knowledge. (Whether it is indispensable remains to be seen, however.) As the very enterprise of logic was historically instigated by the success of the Euclidean axiom system, the notion of implication as developed and investigated in that discipline traditionally amounts to implication-without-exceptions. Implication-as-a-rule, that is, with possible exceptions, as defined and investigated in this thesis, is mainly proposed as a suitable alternative to gain insight into the nature of practical reasoning.

§3 Practical Argumentation

Other motivations for the study of non-monotonic logic in general, and for the contents of this thesis in particular, have to do with the *purpose* of logic. Logic investigates human reasoning, by analyzing and classifying arguments. The only purpose of such an investigation is that it should help us to distinguish good (correct) reasoning from bad (incorrect) reasoning. For mathematical reasoning, a formalism like predicate logic is quite suitable. However, reasoning in everyday life is typically non-mathematical (which is definitely not synonymous with "incorrect"), and the study of logic should, if possible, be an aid in such practical argumentation as well. As such, the investigation of non-monotonic formalisms might be of greater help than the study of monotonic formalisms, such as predicate logic. The principle of monotony, as suitable as it may be for mathematical reasoning, is a misleading principle in practical reasoning, as will be shown by some examples, below.

Objections to the principle of monotony are often related to analogous objections to the principle of *transitivity*: if a implies b, and b implies c, then a implies c. Or, equivalently, to the assumption that any chain of acceptable arguments is automatically acceptable as a whole. A very old example of practical argumentation conflicting with the principle of transitivity is the famous Sorites paradox, or paradox of the sandheap: any sandheap remains a sandheap after removing one grain of sand. The remaining sandheap shares this property. Hence every sandheap remains a sandheap after removing 2 grains of sand. Continuing this chain of arguments leads to the conclusion that it does not take sand at all to have a sandheap. The paradox is caused, of course, by the fact that the notion of a sandheap, as intended here, is vague. But, since (practical) argumentation may involve such vague notions, the sandheap shows that the principle of transitivity is not generally applicable. It does not show, however, that the principle of monotony is not generally applicable.

Another example where the transitivity rule fails is the example about birds and penguins in §2. In general, the transitivity rule, "a $\rightarrow$ b, b $\rightarrow$ c implies a $\rightarrow$ c," can fail if "b" denotes an ambiguous sentence or a sentence involving abstract notions (see §2). That is, if "b" means something different in "a $\rightarrow$ b" than in "b $\rightarrow$ c." This latter explanation might also account for the following example: "if there is both sugar and diesel oil in my tea, then there is sugar in it" and "if there is sugar in my tea, then I like it" do not imply "if there is both sugar and diesel oil in my tea, then I like it." In contrast with the Sorites paradox, these examples (and the ones to follow) are directly connected to failure of the principle of monotony, since they show that, for some sentences a, b, c, "b $\rightarrow$ c" does not imply "(a & b) $\rightarrow$ c."

The principle of transitivity can also fail if "$\rightarrow$" denotes something different in "a $\rightarrow$ b" than in "b $\rightarrow$ c." For example, "if the grass is wet, then the sprinkler has been on" and "if it has rained, then the grass is wet" do not imply "if it has rained, then the sprinkler has been on," nor "if it has rained and the grass is wet, then the sprinkler has been on." This example does not involve two different meanings of "the grass is wet." It involves a distinction between evidential implication and causal implication (the grass being wet is evidence for, but not the cause of the sprinkler having been on, while the

rain caused the grass to be wet). In this case, however, one could also say that, although the assumptions were phrased as "$a \rightarrow b$, $b \rightarrow c$," their content actually amounts to "$a \rightarrow b$, $b \leftarrow c$," which does not imply "$a \rightarrow c$," nor "$(a \& b) \rightarrow c$," nor "$(a \& b) \leftarrow c$."
Examples involving *possibility* or *permission* form a category of their own, involving subtleties concerning the meaning of implication. For example, there is a difference between "if x is a natural number, then it is possible that $x = 0$" and "for every natural number x, it is possible that $x = 0$." In a somewhat more disguised form, imagine a patient who reasons as follows. "My brother had disease X, and was cured by medicine M. Hence, it is possible to cure X with medicine M. I have got disease X. Hence, it is at least possible to cure me by medicine M. That is, if my doctor claims he is sure that I cannot possibly be cured by medicine M, he is ignorant, or just stubborn." The doctor, on the other hand, knows of two varieties of disease X, X_1 and X_2. X_1 is curable with medicine M, X_2 is not, and the patient has X_2. In simplistic terms, this example shows that "X is curable" and "my disease is X" does not imply "my disease is curable," because of the special nature of a notion like "curable." The example about Fred's pyjamas in §2 also involves such a notion, since the relation "... fits into ..." amounts to "... somehow fits into ...", or "...*possibly* fits into ...". An other example involves "permission" instead of "possibility": "You may only kill people that are guilty of adultery" implies "In principle, you may, in some circumstances, kill other people." This latter sentence, taken in isolation, could be used to say "You may kill someone who stole fl 25,– from you," although this conclusion is by no means authorized by the original rule.
Most mental errors in practical reasoning (and in intellectual writing) seem to have a common source that is hard to define. It can be described as (unjustified) reasoning from conclusions previously drawn, ignoring the origin of those conclusions. Or, alternatively, as drawing conclusions by looking at words only, without considering their original intention. The examples above point, in particular, to the principle of transitivity as misleading our judgment in practical reasoning. Sometimes we are not misled, because we are too familiar with the subject (as in the rain and sprinkler example). But sometimes the principle seduces us even on subjects we are quite familiar with (such as the last example). Note that, in these examples it is *applying* the principle of monotony (or transitivity) that would constitute a *logical* error, in contrast with the examples from the beginning of §1, where non-monotony arose from the acceptance of logically unsound behaviour as the intended behaviour. As said, we are interested in such types

of "correct" non-monotonic reasoning, since we want to emphasize that it is possible to think of the rule of monotony as an unsound principle of reasoning.

But this is not the only motivation for non-monotonic logic resulting from reflection on the purpose of logic. An investigation of reasoning, if it is to be of any help in understanding practical reasoning, will require some analysis of the notion of *implication*. In practical reasoning, there is a variety of possible interpretations of implication, the "classical" interpretation being only one of them. For example, sentences like "if the water boils, then turn out the gas" or "if you turn out the gas, then the water will stop boiling" denote some connection between actions and statements. That even the latter statement is not covered by the usual interpretation of implication is seen by the fact that it involves a notion of implication that does not satisfy the principle of monotony: "if you turn out the gas, and shortly afterwards you turn it back on again, then the water will not stop boiling." Although it certainly is possible to interpret such statements using ordinary implication, they usually have another intention. Likewise, a sentence like "if the price increases, sales will decrease" does not amount to "in every possible world in which the price is higher than in the actual world, sales are lower than in the actual world." Capturing the content of such sentences requires other constructions.
Several researchers have elaborated alternative interpretations of implication as used in everyday life. In conditional logic, for example, (see [Lewis 73]) an attempt is made to find semantic, mathematical constructions that capture the notion of implication involved in sentences like "if Caesar had been in charge in Yugoslavia, he would have used the atomic bomb" or "if Caesar had been in charge in Yugoslavia, he would have used catapults." Although conditional logic is usually said to investigate "counterfactual" conditional statements, the essential point seems to be to provide an acceptable mathematical treatment of ceteris-paribus-implication: "if a (*and all "other" things remain unchanged*), then b" (see [Nieuwint 90]). Adams (see [Adams 75]) used ideas from probability theory to interpret and investigate "plausible implication": "if a, then it is highly likely that b." In discussing the alternative treatments of implication, we should also mention Grice's discussion of "conversational implicature" (see [Grice 89]), a linguistic, non-mathematical approach. These investigations, though rather diverse, have one thing in common: each of the approaches leads to some form of non-monotony.

There is more to be learned from these efforts, however. A closer look at the constructions arouses the following suspicion: implication as used in practical reasoning might have the same character as the abstract notions in §2: any mathematical construction capturing the intention of a particular implicational statement might need precization if that statement is used for other purposes. It is quite possible, then, that there is no mathematical formalism that manages to capture all the subtle differences in meaning (of implication) that occur in practical reasoning. And even if we could develop such a system, it would be too complicated to use, since it would force us to be unreasonably specific about our intentions, whenever we formulate an implicational statement.

The approach followed in this thesis is an attempt to overcome this difficulty. Superficially, however, it could also be described as the investigation of just another non-standard interpretation of implication, named "implication-up-to-possible-exceptions," emphasizing the following aspect of implication. In reasoning in general, the purpose of implicational statements is not to guarantee the truth of its conclusion whenever certain conditions are fulfilled, as is the purpose of theorems in mathematics. Implication serves as an instrument by which humans organize (some of) their real-world knowledge. Implication-without-exceptions is a powerful instrument in mathematical reasoning, but it is hardly applicable anywhere else. Implication which is to be taken seriously unless we have a good reason not to, should be much better in this respect. However, claiming that it is not the purpose of arguments to acquire absolute certainty is something different than claiming that it is not the purpose of logic to distinguish between correct and incorrect reasoning.

§4 Plan

To recapitulate, there is a variety of motivations to develop and investigate non-monotonic formalisms, and different motivations will typically lead to essentially different formalisms. In this thesis, we are interested in non-monotony as an important aspect of practical reasoning. In particular, we are interested in implication as used in practical reasoning, and we restrict our attention to only one aspect of it, namely the fact that a valid implicational statement may have exceptions (but nevertheless be valid). It is to be understood that this subject is related to, but nevertheless different from investigating conditional-sentences-as-used-in-natural-language. In fact, we will completely avoid linguistic issues.

Practical reasoning seems to be not susceptible to precise formalization, at least not in the same way as mathematical reasoning. Hence, the formalisms we will provide are claimed to be no more than caricatures of practical reasoning, in about the same way as classical logic can be seen as a caricature of practical reasoning. It is not our aim to come up with some formalism to be directly applied in intelligent computer software; the results should be seen as attempts to gain insight into the nature of practical argumentation.

In Chapter 2, we will present a topological interpretation of implication-with-possible-exceptions, inspired by geometrical intuitions. This semantic construction is not intended to bear direct resemblance to any intuitive interpretation of natural implication, it serves as basic material for the rest of the thesis. The simplest way to use this definition in a caricature of practical reasoning, however, turns out to be equivalent to the well-known approaches of Shoham or Kraus et al. (see [Shoham 88] and [KLM 90]), involving partially ordered sets of possible worlds.

In Chapter 3, the same topological notion will be used to interpret inference. This allows us to talk about rules of inference (that is, laws of logic) as rules-with-possible-exceptions. It is then established that, in at least one natural extension of the simple system of Chapter 2, each of the laws of classical logic is valid-as-a-rule, including the rule of monotony. The system, however, is non-monotonic from a mathematical point of view (since the rule of monotony has exceptions).

In Chapter 4, the topological notion of Chapter 2 will be used to interpret universal quantification, which will be used in Chapter 5 to provide another way to think of rules of inference as rules-with-possible-exceptions. The advantage of this system over the one in Chapter 3 is, among other things, that it satisfies "monotony of independent additions." For example, if a, b, c are three distinct basic formulas, then $\dfrac{a \rightarrow b,\ a,\ c}{b}$ is valid, but $\dfrac{a \rightarrow b,\ a,\ \neg b}{b}$ is not.

Finally, there are two appendices. Appendix A contains standard material of propositional logic, Appendix B contains the necessary standard definitions used in elementary topology.

Chapter 2
Full Subsets

Inspired by geometrical ideas, we provide a mathematical notion, and use it to define implication-up-to-possible-exceptions. The resulting semantics of non-monotonic logic turns out to be comparable to semantics using partial orderings.

§5 The Calculus of Degenerate Cases

Euclidean geometry is an old discipline of mathematics. Most probably the oldest one that requires a notion of implication, in that it is organized as a small number of assumptions, called axioms, and an unlimited number of theorems, each being logically implied by the axioms. Moreover, a typical theorem or axiom is in itself an implicational statement, stating about a number of objects that they will have certain properties whenever they satisfy some conditions. The interesting feature, for us, is the phenomenon of theorems having *degenerate cases*. It is a phenomenon commonly seen in Euclidean geometry, although it is usually not considered important. Some mathematicians find it amusing, others annoying. Let us give an example.

5.1 Theorem Let ℓ be a line in a plane, P a point in the same plane. Then, in that plane, there is at most one line through P perpendicular to ℓ.

If P is not on ℓ, the proof could be something like the following.

Suppose that m and m' are lines through P, each one perpendicular to ℓ. Let σ denote line-reflection in line ℓ. Then $\sigma(m) = m$, $\sigma(m') = m'$. Since P is not on ℓ, $\sigma(\mathrm{P}) \neq \mathrm{P}$. But $\sigma(\mathrm{P})$ is on m, as well as on m'. Hence m and m' have two different points in common. Therefore, $m = m'$, which proves the conclusion.

Although the conclusion of Theorem 5.1 also holds if P is on ℓ, the idea of the proof above cannot possibly be used in that case. Now the case "P is on ℓ" is called a degenerate case.

Some people would think of this example as being just a matter of proof-by-cases. But there is more to it. The first is, that there are also theorems having degenerate cases for which the theorem fails. It is a good habit in mathematics to take care of such cases in the wording of the theorem in question, namely by adding some extra conditions. For example:

5.2 Theorem Let ℓ be a line (in Euclidean space), P a point (in the same space), *such that P is not on* ℓ. Then (in that space) there is at most one line through P perpendicular to ℓ.
(Proof: see above.)

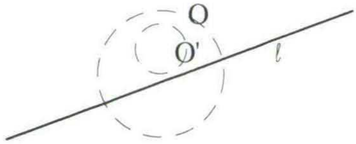
But this habit veils an essential practical aspect, namely the fact that even experienced mathematicians remember such theorems or proofs as valid up to some exceptional cases, the latter being treated separately, for safety. For this kind of geometrical theorems, the possible degenerate cases are easily traceable; for example, by carefully following the details of a provisional proof, like the one above. In other fields of human knowledge this typically is not so easy, if it is possible at all.
Furthermore, even in the case of Theorem 5.1, where both cases do satisfy the conclusion, the case "P is on ℓ" is said to be degenerate, while the case "P is not on ℓ" is not. This peculiar asymmetry is explained, in this case, by the intuitive idea that for "almost all" lines ℓ and "almost all" points P, P is not on ℓ. In general, there is a number of other statements like this. For example, "for almost all points P and Q: P is distinct from Q", "given a line, almost all lines are not perpendicular to that line", "given a line, almost all points are not on that line", etc., etc. One might ask whether the intuitive content of such sentences can be captured by some mathematical definition.
In fact, there are several possibilities. Using elementary topological ideas, there is one option with particularly nice properties, as we will see. The definitions that we will use, here, are not claimed to capture the intuitive ideas in every respect, but for the purposes at hand, they will suffice. Let us, first, restrict our attention to statements of the simplest kind. Given some subset of the plane, what does it mean to say "almost all" points (in the plane) are in that subset?

5.3 Definition Let X be a topological space (see Appendix B), and let a be a subset of X. We say that a is *full* (in X) whenever every nonempty open O contains a nonempty open O' (open in X) with $O' \subseteq a$.

(As said in Appendix B, we will use the variables O, O_1, O' etc., *exclusively*, to denote open sets in some topological space. Consequently, we will skip the word "open" as much as possible.)

5.4 Example If E denotes the Euclidean plane, and ℓ a line in it, then $E \setminus \ell$ is full in E ("almost all elements of E are elements of $E \setminus \ell$.")

The following picture might suffice to see this:



5.5 Proposition Let X be a topological space and $a \subseteq X$. Then the following statements are equivalent:

i) a is full in X,

ii) the interior of a is dense in X,

iii) a contains a subset that is both open and dense in X,

iv) for every nonempty O, a is dense in O and a^c is not,

v) a^c is nowhere dense in X.

(Provable by elementary check.)

The intuitive content of the notion of a full subset is the following: a set is full in X whenever its complement (in X) has got holes wherever we look (in X) and the set itself does not have such holes (cf. 5.5 iv, above). Note that, in the space $\mathbb{R}$ of real numbers, $\mathbb{Q}$ is dense, but not full, and $\mathbb{Q}^c$ is also dense but not full (every nonempty open interval contains both rational and irrational numbers). In every topological space, any full subset of X is dense in X, but not vice versa. Any set which is both open and dense in X is full in X, but not vice versa. The following properties are crucial:

5.6 Proposition

If X is a topological space, and a and b are subsets of X, then:

i) If a and b are full in X, then $a \cap b$ is full in X,

ii) If a is full in X and $a \subseteq a' \subseteq X$, then a' is full in X,

iii) If a is full in X, then a^c is not full in X, unless $X = \varnothing$.

Proof:

i) is a trivial consequence of 5.5 iii) and the fact that the intersection of any pair of open and dense sets is open and dense.

ii) is a trivial consequence of Definition 5.3.

iii) follows from i) above and the fact that $\varnothing$ is not dense in X, unless $X = \varnothing$.

□

Since every subset of a topological space is canonically equipped with an induced topology, it is straightforward to extend Definition 5.3.

5.7 Definition Given a topological space X and $a, b \subseteq$ X, we say that *b is full in a* (written as "$a \to b$") whenever $a \cap b$ is a full subset of a-with-induced-topology.

"$a \to b$" is considered to be a possible interpretation of the phrase "almost all elements of a are also in b".

5.8 Proposition If X is a topological space and $a, b \subseteq$ X, then $a \to b$ (only) if every O (open in X) such that $O \cap a \neq \varnothing$ contains an O' such that $O' \cap a \neq \varnothing$ and $O' \cap a \subseteq b$.
(Provable by elementary check.)

5.9 Example If E denotes the Euclidean plane, and ℓ some line in it, then ℓ^c is full in E, but not in ℓ. After all, almost all elements of E are in $E \setminus \ell$. But of the elements of ℓ, not even a single one is in $E \setminus \ell$. And $\ell \to \varnothing$ is not true, by Proposition 5.6 iii).

The notion of a full subset can also be used to interpret more general sentences, like "for almost all points P, Q: P is distinct from Q", by using the product topology on $E \times E$ (where E denotes the Euclidean plane). That issue is postponed, however, until Chapter 4.
The notion defined in 5.7 could be seen as a topological notion capturing the intuition of inclusion-up-to-possible-exceptions. It has very nice properties. Example 5.9 shows that it is non-monotonic: $E \to \ell^c$ is true, but $E \cap \ell \to \ell^c$ is not true; hence, $a \to b$ does, in general, not imply $a \cap c \to b$. On the other hand, we have :

5.10 Proposition For every topological space, and for all $a, b, c \subseteq$ X :

i) $a \to a$
ii) $a \to b$ (only) if $a \to a \cap b$
iii) if $a \to b$, $a \to c$ then $a \to b \cap c$
iv) if $a \to b$ and $b \subseteq c$ then $a \to c$
v) if $a \to \varnothing$ then $a = \varnothing$
vi) if $a \to c$ and $b \to c$ then $a \cup b \to c$

Proof: i) and ii) are trivial. iii), iv) and v) are direct consequences of Proposition 5.6 i), ii) and iii), respectively.

To prove vi): Suppose that $a \to c$ and $b \to c$.

Suppose that $O \cap (a \cup b) \neq \varnothing$. Say $O \cap a \neq \varnothing$.

Let $O' \subseteq O$ be such that $O' \cap a \neq \varnothing$ and $O' \cap a \subseteq c$.

If $O' \cap b = \varnothing$, then $O' \cap (a \cup b) \neq \varnothing$ and $O' \cap (a \cup b) \subseteq c$.

If $O' \cap b \neq \varnothing$, then let $O'' \subseteq O'$ be such that

$O'' \cap b \neq \varnothing$ and $O'' \cap b \subseteq c$.

Then $O'' \cap (a \cup b) \neq \varnothing$, and $O'' \cap a \subseteq O' \cap a \subseteq c$, hence

$O'' \cap (a \cup b) \subseteq c$.

Hence, $a \cup b \to c$.

□

More important are the following properties:

5.11 Proposition For every topological space, and for all $a, b, c \subseteq X$:

i) if $a \to b$ and $a \to c$ then $a \cap b \to c$,

ii) if $a \to b$ and $a \cap b \to c$ then $a \to c$.

Proof: i) Suppose that $a \to b$ and $a \to c$.

For every O such that $O \cap a \cap b \neq \varnothing$, let $O' \subseteq O$ be such that $O' \cap a \neq \varnothing$ and $O' \cap a \subseteq b$. Let $O'' \subseteq O'$ be such that $O'' \cap a \neq \varnothing$ and $O'' \cap a \subseteq c$.

Then $O'' \cap (a \cap b) = O'' \cap a \neq \varnothing$ and $O'' \cap (a \cap b) \subseteq c$.

Hence, $a \cap b \to c$.

ii) Suppose that $a \to b$ and $a \cap b \to c$.

For every O such that $O \cap a \neq \varnothing$, let $O' \subseteq O$ be such that $O' \cap a \neq \varnothing$ and $O' \cap a \subseteq b$. Then $O' \cap (a \cap b) \neq \varnothing$.

Let $O'' \subseteq O'$ be such that

$O'' \cap (a \cap b) \neq \varnothing$ and $O'' \cap (a \cap b) \subseteq c$.

Then $O'' \cap a \neq \varnothing$ and $O'' \cap a \subseteq O'' \cap (a \cap b) \subseteq c$.

Hence, $a \to c$.

□

5.12 Proposition Let X be a topological space and $a, b, c \subseteq X$. Then :

i) $a \subseteq b$ implies $a \to b$, but, in general, $a \to b$ does not imply $a \subseteq b$,

ii) if $a \leftrightarrow b$ (i.e., $a \to b$ and $b \to a$),
then $a \to c$ implies $b \to c$, but $c \to a$ does not imply $c \to b$,

iii) $a \to b, b \to c$ does not imply $a \to c$, but it does if $a \supseteq b \supseteq c$,

iv) $a \cap b \to c$ implies $a \to b^c \cup c$, but not vice versa,

v) $a \to b$ does not imply $b^c \to a^c$,

vi) $a \subseteq b \subseteq c$, $c \to a$ implies $b \to a$.

Proof: The positive statements are easy corollaries of Propositions 5.10 and 5.11. For example, to prove iv), note that, by 5.10 iv), $a \cap b \to c$ implies $a \cap b \to b^c \cup c$ and, together with $a \cap b^c \to b^c \cup c$ (5.10 i), this implies $a \to b^c \cup c$, by 5.10 vi). vi) is proved as follows : $a \subseteq b \subseteq c$ and $c \to a$ implies $c \to b$, by 5.10 iv), which implies $b = c \cap b \to a$, by 5.11 i).
For the negative statements, let E denote the Euclidean plane, and let ℓ be a line in E. Then $\mathrm{E} \to \ell^c$ and $\ell \neq \varnothing$, hence $\ell \not\to \varnothing$, hence $\ell \not\to \ell^c$. Hence:

i) $\mathrm{E} \to \ell^c$, $\mathrm{E} \not\subset \ell^c$,

ii) $\mathrm{E} \leftrightarrow \ell^c$, $\ell \to \mathrm{E}$, $\ell \not\to \ell^c$,

iii) $\ell \to \mathrm{E}$, $\mathrm{E} \to \ell^c$, $\ell \not\to \ell^c$,

iv) $\mathrm{E} \to \ell^c \cup \varnothing$, $\mathrm{E} \cap \ell \not\to \varnothing$,

v) $\mathrm{E} \to \ell^c$, $\ell \not\to \varnothing$.

□

5.13 Definition A topological space X is called *monotonic* whenever, for all $a, b, c \subseteq \mathrm{X}$, $a \to b$ implies $a \cap c \to b$.

As seen before, the Euclidean plane is a non-monotonic space.
On the other hand, if X is equipped with the minimal topology (i.e., $\varnothing$ and X itself are the only open subsets of X), then X is a monotonic space.
(Proof : Using Proposition 5.8, the definition of $a \to b$ ("every O such that $\mathrm{O} \cap a \neq \varnothing$ contains an O' such that $\mathrm{O}' \cap a \neq \varnothing$ and $\mathrm{O}' \cap a \subseteq b$") amounts, in this case, to "if $\mathrm{X} \cap a \neq \varnothing$, then $\mathrm{X} \cap a \subseteq b$", which is equivalent to "$a \subseteq b$".)
Likewise, if X is equipped with the discrete topology (i.e., all subsets of X are open), then X is a monotonic space.
(Proof : In this case, the definition of $a \to b$ implies "for every $\mathrm{x} \in \mathrm{X}$ such that $\{\mathrm{x}\} \cap a \neq \varnothing$, $\{\mathrm{x}\} \cap a \subseteq b$", which also is equivalent to "$a \subseteq b$".)

5.14 Proposition Let X be a topological space. Then the following statements are equivalent :

i) X is monotonic

ii) for all $a, b \subseteq \mathrm{X}$, $a \to b$ implies $a \subseteq b$,

iii) for all $a \subseteq \mathrm{X}$, $\mathrm{X} \to a$ implies $\mathrm{X} = a$,

iv) every open set of X is closed in X.

Proof: i) $\Rightarrow$ ii): If X is monotonic, then $a \to b$ implies $a \cap \{\mathrm{x}\} \to b$, for all $\mathrm{x} \in \mathrm{X}$. Hence, it implies $\{\mathrm{x}\} \to b$, for all $\mathrm{x} \in a$.

ii) $\Rightarrow$ iv): Every open set is full in its closure (see Definition B.7). Hence, if X satisfies ii), then every open set equals its own closure.
iv) $\Rightarrow$ iii): If $X \to a$, then (according to 5.5 iii) there is an O, open in X such that $O \subseteq a$ and O is dense in X. By iv), O is closed, hence $O = X = a$.
iii) $\Rightarrow$ ii): If $a \to b$ and $a \not\subset b$, then $X = a^c \cup a \to a^c \cup b$, and $X \neq a^c \cup b$.
ii) $\Rightarrow$ i) is trivial.

□

As a matter of fact, it is easy to see :

5.15 Proposition For every topological space, X, the following statements are also equivalent :

i) X is monotonic,
ii) for all $a, b, c \subseteq X$, $a \to b$ and $b \to c$ implies $a \to c$,
iii) for all $a, b \subseteq X$, $a \to b$ implies $b^c \to a^c$,
iv) for all $a, b, c \subseteq X$, $a \to b^c \cup c$ implies $a \cap b \to c$.

Proof:
i) $\Rightarrow$ ii), i) $\Rightarrow$ iii) and i) $\Rightarrow$ iv) are trivial consequences of 5.14 i) $\Leftrightarrow$ ii).
To prove i) $\Leftarrow$ ii), i) $\Leftarrow$ iii) and i) $\Leftarrow$ iv):

Let X be a non-monotonic space.
By 5.14 i) $\Leftrightarrow$ iii), there is a $d \subseteq X$ such that
$X \to d$ is true, but $X = d$ is not.
Then $d^c \to X$, $X \to d$, $d^c \not\to d$ (hence, ii) is not true),
$X \to d$, $d^c \not\to \varnothing$ (hence, iii) is not true),
and $X \to d \cup \varnothing$, $X \cap d^c \not\to \varnothing$ (hence, iv) is not true).

□

5.16 Proposition ("Loop-rule")

Let X be a topological space. Then, for all $a, b, c \subseteq X$,
if $a \to b$, $b \to c$, and $c \to a$, then $a \to c$, (hence) $a \leftrightarrow b$, $b \leftrightarrow c$, etc.
Proof: Suppose that $a \to b$, $b \to c$, and $c \to a$.
For every O such that $O \cap a \neq \varnothing$, there is an $O' \subseteq O$ such that

$O' \cap a \neq \varnothing$ and $O' \cap a \subseteq b$.
Hence, $O' \cap b \neq \varnothing$ and there is an $O'' \subseteq O'$ such that
$O'' \cap b \neq \varnothing$ and $O'' \cap b \subseteq c$.
Hence, there is an $O''' \subseteq O''$ such that $O''' \cap c \neq \varnothing$ and $O''' \cap c \subseteq a$.
Now, $O''' \cap c \subseteq O''' \cap a \subseteq O''' \cap b \subseteq O''' \cap c$.
Hence, $O''' \subseteq O$, $O''' \cap a \neq \varnothing$ and $O''' \cap a \subseteq c$.

Hence, $a \to c$. The rest is seen by symmetry.

□

The following three lemmas will turn out to be particularly useful.

5.17 Lemma ("the open-lemma")
For all $a, b, c \subseteq X$, $a \to b$ implies $a \cap c \to b$ if c is open (either in X or in a).
Proof: If $a \to b$ and c is open (either in X or in a), then for every O such that $(O \cap c) \cap a \neq \varnothing$, there is an $O' \subseteq (O \cap c)$ such that

$$O' \cap a \neq \varnothing \text{ and } O' \cap a \subseteq b.$$

Then $O' \cap (a \cap c) = O' \cap a \neq \varnothing$ and $O' \cap (a \cap c) \subseteq b$.
Hence, $a \cap c \to b$.

□

5.18 Lemma ("the dense-lemma")
For all $a, b, c \subseteq X$, $a \to b$ implies $a \cap c \to b$ if c is dense in a.
Proof: If $a \to b$ and c is dense in a, then for every O such that

$$O \cap c \cap a \neq \varnothing \quad (\text{hence, } O \cap a \neq \varnothing),$$

there is an $O' \subseteq O$ such that

$$O' \cap a \neq \varnothing \text{ and } O' \cap a \subseteq b.$$

Since c is dense in a, this implies $O' \cap a \cap c \neq \varnothing$. But $O' \cap a \cap c \subseteq b$.
Hence, $a \cap c \to b$.

□

5.19 Lemma ("the closed-lemma")
For all $a, b, c \subseteq X$, $a \to b$ implies $a \cap c \to b$ if b is closed (either in X or in a).
Proof: If b is closed (either in X or in a), and $a \to b$, then b is both closed in a and dense in a. Hence, $a \subseteq b$. Hence, for all c, $a \cap c \to b$.

□

§6 Application

Definition 5.7 could be said to capture the notion of inclusion-up-to-possible-exceptions. However, our original aim was the study of practical reasoning by providing a non-standard interpretation of implication, namely implication-with-possible-exceptions. The simplest way to use Definition 5.7 for that purpose is the following.
The usual way to represent, in mathematical terms, the meaning of an implicational statement "a implies b" is to associate it with a collection of pos-

sible worlds, in each of which a and b are either true or false. The statement a is said to imply the statement b whenever all possible worlds that satisfy a do also satisfy b. On this point, we will just adhere to usage. However, by using the topological interpretation of inclusion-up-to-possible-exceptions, as defined in 5.7, we can now interpret implication-up-to-possible-exceptions in a similar way, if the collection of possible worlds is provided with a topology.

6.1 Definition Let B be a finite Boolean algebra (see Appendix A).
A *topological model* (of B) is a pair (X, φ), where

i) X is a topological space,

ii) $\varphi: B \to \mathcal{P}(X)$ is a Boolean translation.

(The elements of X are nicknamed "possible worlds", and for all $a \in B$, $\varphi(a)$ is the set of all possible worlds in which a is said to be true.)

For $a, b \in B$, we will say "$a \to b$ is *true in* (X, φ)" whenever $\varphi(a) \to \varphi(b)$ is true in the topological space X, that is: *almost* all possible worlds in X that satisfy a do also satisfy b.

For example, let B be the free Boolean algebra generated by two (distinct) basic formulas, a and b. Then there exist topological models of B in which $a \to b$ is true, but $(a \wedge \neg b) \to b$ is not.
(Proof: If E is the Euclidean plane and ℓ a line in E, then there is one and only one Boolean translation $\varphi: B \to \mathcal{P}(E)$ such that

$\varphi(a) = E$ and $\varphi(b) = E \setminus \ell$.

In the topological model (E, φ), $a \to b$ is true (since $E \to \ell^c$),
while $(a \wedge \neg b) \to b$ is not : $E \cap \ell \to \ell^c$ is not true, since ℓ is not empty.)
The intuitive content of the possible worlds, as well as of the topology, may remain unspecified, the only relevant aspect of a topological model being the combinations of sentences "$s \to t$" (for $s, t \in B$) it makes true. Moreover, the term "world" should not be taken too literally. For example, we could think of the formulas a and b as signifying the sentences "x is a bird" and "x can fly", respectively. Then the elements of X should rather be thought of as possible instances that x could refer to. It is not claimed, of course, that the set of all birds bears any resemblance with the Euclidean plane, but that the relation between birds and birds that cannot fly could be thought to resemble the relation between E and ℓ.
Note that, for this reading of a and b, the topological model above shows that the statement "if x is a bird, then it can fly" does not imply "if x is a bird

and x cannot fly, then x can fly". According to classical logic, however (that is, interpreting implication as implication-without-exceptions), this latter statement is a necessary consequence of the former. In this way, certain combinations of sentences considered inconsistent when using the more usual interpretation of implication are nevertheless representable using topological models. We will call them "topologically representable" or even "topologically consistent".

Let us give some more examples. In each example, B is assumed to be the free Boolean algebra generated by the basic formulas occurring in it. E denotes the Euclidean plane. ℓ, m etc. denote lines in E, while P, Q etc. denote points in E.

6.2 Example (Exceptions to exceptions)

The combination of the assumptions "If x is a bird, then x can fly", "If x is a penguin, then x cannot fly" and "If x is a yellow penguin, then x can fly" does not imply "If x is a penguin, then x can fly", nor "If x is a yellow penguin, then x cannot fly". That is:

$$\mathtt{b} \to \mathtt{f},\ \mathtt{b} \wedge \mathtt{p} \to \neg\mathtt{f},\ \mathtt{b} \wedge \mathtt{p} \wedge \mathtt{y} \to \mathtt{f}$$

do not imply $\mathtt{p} \wedge \mathtt{b} \to \mathtt{f}$, nor $\mathtt{p} \wedge \mathtt{b} \wedge \mathtt{y} \to \neg\mathtt{f}$.

Proof: Let P be a point on a line, ℓ, and φ a Boolean translation $B \to \mathcal{P}(E)$ such that $\varphi(\mathtt{b}) = E$, $\varphi(\mathtt{f}) = \ell^c \cup \{P\}$, $\varphi(\mathtt{p}) = \ell$, $\varphi(\mathtt{y}) = \{P\}$. Then it is easy to check that the topological model (E, φ) falsifies the implication.

□

6.3 Example (More specific rules)

The assumptions "If x is a student, then x is unemployed", "If x is an adult, then x is employed" do not imply "If x is an adult and x is a student, then x is employed", nor "If x is an adult and x is a student, then x is *un*employed". However, together with the assumption "If x is a student, then x is an adult", they imply the latter, but not the former. In symbols:

$\mathtt{a} \to \mathtt{c},\ \mathtt{b} \to \neg\mathtt{c}$ does not imply $\mathtt{a} \wedge \mathtt{b} \to \mathtt{c}$, nor $\mathtt{a} \wedge \mathtt{b} \to \neg\mathtt{c}$,

while $\mathtt{a} \to \mathtt{c},\ \mathtt{b} \to \neg\mathtt{c},\ \mathtt{a} \to \mathtt{b}$ implies $\mathtt{a} \wedge \mathtt{b} \to \mathtt{c}$, but not $\mathtt{a} \wedge \mathtt{b} \to \neg\mathtt{c}$.

Proof: Let ℓ be a line and φ a Boolean translation $B \to \mathcal{P}(E)$ such that $\varphi(\mathtt{a}) = E$, $\varphi(\mathtt{b}) = \ell$, $\varphi(\mathtt{c}) = \ell^c$.

Then $\mathtt{a} \to \mathtt{c}$ and $\mathtt{b} \to \neg\mathtt{c}$ are true in (E, φ), but $\mathtt{a} \wedge \mathtt{b} \to \mathtt{c}$ is not.

On the other hand, if $\varphi(\mathtt{a}) = \ell$, $\varphi(\mathtt{b}) = E$ and $\varphi(\mathtt{c}) = \ell$, then $\mathtt{a} \to \mathtt{c}$ and $\mathtt{b} \to \neg\mathtt{c}$ are also true, but $\mathtt{a} \wedge \mathtt{b} \to \neg\mathtt{c}$ is not.

Note that in this latter topological model, $\mathtt{a} \to \mathtt{b}$ is also true.

Hence, $\mathtt{a} \to \mathtt{c},\ \mathtt{b} \to \neg\mathtt{c},\ \mathtt{a} \to \mathtt{b}$ does not imply $\mathtt{a} \wedge \mathtt{b} \to \neg\mathtt{c}$.

However, $a \to c$, $a \to b$ implies $a \wedge b \to c$, by Proposition 5.11 i).

□

6.4 Example (Conflicting arguments)

$a \to c$, $b \to c$ does not imply $a \wedge b \to c$.

Proof: A line, ℓ, devides the plane in two halfplanes, say H_1 and H_2, each containing ℓ. Let φ be a Boolean translation $B \to \mathcal{P}(E)$ such that $\varphi(a) = H_1$, $\varphi(b) = H_2$, $\varphi(c) = \ell^c$. Then $a \to c$ and $b \to c$ are true in (E, φ), but not $a \wedge b \to c$. Note that $a \to \neg b$ and $b \to \neg a$ are also true in (E, φ).

□

For example, if a murder has been committed in Johannesburg, and John was seen in London at the time of the murder, he has got a valid alibi. Likewise, if John was seen in Paris. But combined, the two alibis would arouse suspicion.

Example 6.4 reflects the phenomenon that, in practical reasoning, providing two arguments (for a single conclusion) might be less convincing than providing only one, in particular when the two arguments contradict each other, or contain conflicting information.

6.5 Example (Irresolute situations)

$a \to b$ does not imply "either $a \wedge c \to b$ or $a \wedge c \to \neg b$".

Proof: Let P be a point on the line ℓ and Q a point not on ℓ. Let φ be a Boolean translation $B \to \mathcal{P}(E)$ such that $\varphi(a) = E$, $\varphi(b) = \ell^c$, $\varphi(c) = \{P,Q\}$. Then $a \to b$ is true in (E, φ), but $a \wedge c \to b$ is not (since $\{P\}$ is open in $\{P,Q\}$ and $P \notin \ell^c$), nor is $a \wedge c \to \neg b$ (since $\{Q\}$ is open in $\{P,Q\}$ and $Q \notin \ell$). Note that $a \to \neg c$ is also true in (E, φ).

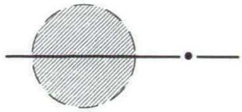
□

For example, if we find the bones of an extinct species of bird, we cannot conclude that that species could fly, solely because that is typically the case for birds. Neither are we in a position to conclude that that species of bird could not fly, however.

6.6 Example

$a \to b$, $a \not\to \neg c$ does not imply $a \wedge c \to b$.

Proof: By choosing coordinates in E, we may identify E with $\mathbb{R}^2$. Let Γ be the disc $\{(x, y) \mid x^2 + y^2 < 1\}$, $P = (2, 0)$, $\ell = \{(x, y) \mid y = 0\}$, a line through P.



Let φ be a Boolean translation $B \to \mathcal{P}(E)$ such that $\varphi(a) = E$, $\varphi(b) = \ell^c$, and $\varphi(c) = \Gamma \cup \{P\}$. Then $a \to b$ is true in (E, φ), but $a \to \neg c$ is not (since Γ is open in E), nor is $a \wedge c \to b$ (since $\{P\}$ is open in $\Gamma \cup \{P\}$ and $P \notin \ell^c$).

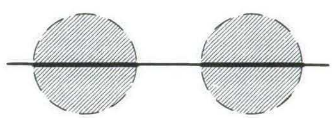
□

We could think of $\{P\}$ as (representing the) penguins, E as birds, Γ as sparrows, ℓ as birds that cannot fly. $\varphi(c)$ then represents the sparrows and penguins together. Then, in our topological model, birds fly, penguins don't, some exceptional sparrows do not fly. The sentence "if x is a bird then x is not a sparrow" is not true, nor is the sentence "if x is a sparrow or a penguin, then x can fly". (Example taken from [Prakken 93].)
Note that $\Gamma \to \ell^c$ is also true in E (sparrows fly), as well as $\Gamma \cup \{(1, 0)\} \to \ell^c$ (if penguins were more like sparrows, then the sentence "if x is either a sparrow or a penguin, then x can fly" would be true).

6.7 Example

$a \to b$ does not imply "either $a \wedge c \to b$ or $a \wedge \neg c \to b$".

Proof: Let Γ_1 and Γ_2 be two discs and ℓ a line like in the picture below.



Let φ be a Boolean translation $B \to \mathcal{P}(E)$ such that $\varphi(a) = \Gamma_1 \cup \Gamma_2$, $\varphi(b) = \ell^c$, and $\varphi(c) = (\Gamma_1 \cap \ell) \cup (\Gamma_2 \setminus \ell)$. Then $a \to b$ is true in (E, φ), but $a \wedge c \to b$ is not (since $\Gamma_1 \cap \ell \to \ell^c$ is not true), nor is $a \wedge \neg c \to b$ (for similar reasons).

□

We could think for example, of Γ_1 as "mammals", Γ_2 as "birds". Birds can fly, mammals cannot fly. Penguins, exceptional birds, cannot fly. Bats, exceptional mammals, can fly. (Example taken from [Prakken 93].)

§7 Preferential Consequence Relations

Now that we have seen some examples, it is time for a complete characterization of all such topologically representable situations or, equivalently, for a complete axiomatization of topological models. In [KLM 90], it was established that all properties mentioned in Propositions 5.10, 5.11, 5.15, and 5.16, as well as the positive results of 5.12, are consequences of only a small number of them, namely: 5.10 i, iv, v and vi, and 5.11 (i and ii). The following notion originates from their article:

7.1 Definition Let $(B, \leq)$ be a Boolean algebra (see Appendix A). A binary relation, $\mid\!\sim$, on B is called a *preferential consequence relation* (on B) whenever

(P1) for all $a \in B$, $a \mid\!\sim a$,

(P2) for all $a, b, c \in B$, if $a \mid\!\sim b$ and $b \leq c$, then $a \mid\!\sim c$,

(P3) for all $a, b, c \in B$, if $a \mid\!\sim b$ and $a \wedge b \mid\!\sim c$, then $a \mid\!\sim c$,

(P4) for all $a, b, c \in B$, if $a \mid\!\sim b$ and $a \mid\!\sim c$, then $a \wedge b \mid\!\sim c$,

(P5) for all $a, b, c \in B$, if $a \mid\!\sim c$ and $b \mid\!\sim c$, then $a \vee b \mid\!\sim c$.

Definition 7.1 is not restricted to finite Boolean algebras. For example, for every topological space X, the binary relation "$\rightarrow$", as defined in 5.7, is a preferential consequence relation on the Boolean algebra $(\mathcal{P}(X), \subseteq)$. (This was proved in Propositions 5.10 and 5.11.) However, the characterization that we will provide will be restricted to preferential consequence relations on finite Boolean algebras, since it is associated with examples from practical reasoning like those in the preceding section, each of which involving finitely many basic formulas. We are going to prove that P1-P5 is a complete set of axioms for problems of that simple kind.

This may seem a little bit strange, since

(P6) for all $a \in B$, if $a \mid\!\sim \perp$, then $a = \perp$

is not a consequence of P1-P5 but does nevertheless hold in every topological space (Proposition 5.10 v). However, P6 is considered redundant, for the following reason.

7.2 Proposition Let $(B, \leq)$ be a Boolean algebra, and $\mid\!\sim$ a preferential consequence relation on B. Then there is a Boolean algebra, B', a Boolean translation φ: B $\rightarrow$ B', and a preferential consequence relation, $\vdash_A$, on B' such that for all a, b $\in$ B,

$$a \vdash b \text{ (only) if } \varphi(a) \vdash_A \varphi(b)$$

and such that $\vdash_A$ satisfies P6 (as well as P1-P5).
(The proof of this proposition is not difficult but somewhat tedious. Therefore, it is skipped.)

To prove that P1-P5/P6 is a complete set of axioms for the relation "$\rightarrow$" in topological models, we use another completeness theorem, originating from [KLM 90]. We will give a number of preliminary definitions, first.

7.3 Definition Let (X, $\leq$) be a partial ordering and a, $b \subseteq$ X.
An element x $\in$ a is *minimal in a* whenever there is no y $\in$ a such that y $\leq$ x except x itself. We will use $a \vdash_{\leq} b$ to denote : every element of a that is minimal in a is an element of b.

If we think of the elements of X as possible worlds, the partial ordering is supposed to describe a "normality" ordering on the possible worlds, or a preference of some possible worlds over others. For $x_1, x_2 \in X$, $x_1 \leq x_2$ is to be read as "x_1 is a more normal world than x_2" or "x_1 is preferred over x_2". Then $a \vdash_{\leq} b$ amounts to: "of all the worlds in a, at least the most normal (preferred) ones are in b".
It is easy to see, that for every *finite* partial ordering (X, $\leq$), the relation $\vdash_{\leq}$ is a preferential consequence relation on the Boolean algebra ($\mathcal{P}$(X), $\subseteq$) and, moreover, that it satisfies P6.

7.4 Theorem (Kraus, Lehmann, Magidor 1990)
Let B be a finite Boolean algebra, and $\vdash$ a preferential consequence relation on B. Then there is a finite partially ordered set (X, $\leq$) and a Boolean translation φ: B $\rightarrow$ $\mathcal{P}$(X) such that

$$\text{for all } a, b \in B,\quad a \vdash b \text{ (only) if } \varphi(a) \vdash_{\leq} \varphi(b).$$

For the proof, the reader is referred to [KLM 90].

We are now ready to prove a similar theorem concerning topological models. The result will be an easy corollary (Corollary 7.6, below) of the following theorem :

7.5 Theorem For every partial ordering, $\leq$, on a finite set X there is at least one topology on X such that, for all $a, b \subseteq X$,

$a \rightarrow b$ (only) if $a \mathrel{|\!\!\sim}_{\leq} b$.

Proof: Let $(X, \leq)$ be a partial ordering.

Define $\tau := \{v \subseteq X \mid \text{if } p \in v \text{ and } q \leq p \text{ then } q \in v\}$. This is a topology on X.

For every $p \in X$, let O_p denote $\{q \in X \mid q \leq p\}$.

Suppose that $a \rightarrow b$. If p is minimal in a, then $O_p \cap a = \{p\}$.

Since $a \rightarrow b$ is true, there is an $O' \subseteq O_p$ such that $O' \cap a \neq \emptyset$ and $O' \cap a \subseteq b$.

Hence, $O' \cap a = \{p\}$ and $p \in b$.

Hence $a \mathrel{|\!\!\sim}_{\leq} b$.

On the other hand, suppose that $a \mathrel{|\!\!\sim}_{\leq} b$. Let O be such that $O \cap a \neq \emptyset$. Say $p \in a$ and $p \in O$. Since X is finite, there is a q that is minimal in a such that $q \leq p$.

For such q : $O_q \subseteq O_p \subseteq O$ and $q \in b$ (since $a \mathrel{|\!\!\sim}_{\leq} b$).

Hence, $O_q \subseteq O$, $O_q \cap a = \{q\} \neq \emptyset$, and $O_q \cap a \subseteq b$.

Hence $a \rightarrow b$.

Hence, for all $a, b \subseteq X$, $a \rightarrow b$ (only) if $a \mathrel{|\!\!\sim}_{\leq} b$.

□

7.6 Corollary / Completeness Theorem

Let B be a finite Boolean algebra, and $\mathrel{|\!\!\sim}$ a preferential consequence relation on B. Then there is a (finite) topological space X and a Boolean translation $\varphi: B \rightarrow \mathcal{P}(X)$ such that

for all $a, b \in B$, $a \mathrel{|\!\!\sim} b$ (only) if $\varphi(a) \rightarrow \varphi(b)$.

(Immediate consequence of Theorems 7.4 and 7.5 above.)

Corollary 7.6 characterizes the relation "$\rightarrow$" in topological models of finite Boolean algebras, thus settling the task of characterizing the "topologically representable situations" as meant in §6. The rest of this section will be spent on clearing up some possibly confusing details and on examining more closely the relationship between topological models and models based on partial orderings.

The first remark to make, is that Theorem 7.4 and Corollary 7.6 can be extended to infinite Boolean algebras. It is unclear, though, whether such an extension is of any use in connection with examples like those of §6, since such puzzles (as well as arguments from practical reasoning) involve only finitely many sentences. However, a finite Boolean algebra may have interesting infinite topological models (for example, on the Euclidean plane, see §6). Hence, we might learn something from the infinitary extension of 7.4, as provided in [KLM 90] :

7.7 Definition

i) Let $(X, \leq)$ be a partial ordering. A subset $a \subseteq X$ is called *smooth* whenever for all $x \in a$, there is a minimal element x' of a such that $x' \leq x$.

ii) Let B be a Boolean algebra. A *preferential model* (of B) is a triple $(X, \leq, \varphi)$ such that:

$(X, \leq)$ is a partial ordering,

φ is a Boolean translation $B \rightarrow \mathcal{P}(X)$,

for every $a \in B$, $\varphi(a)$ is smooth in $(X, \leq)$.

7.8 Proposition (Kraus, Lehmann, Magidor, 1990)
Let B be a (not necessarily finite) Boolean algebra, and let $\mid\!\sim$ be a binary relation on B. Then $\mid\!\sim$ is a preferential consequence relation (only) if there is a preferential model $(X, \leq, \varphi)$ of B such that,

for all $a, b \in B$, $\varphi(a) \mid\!\sim_{\leq} \varphi(b)$ (only) if $a \mid\!\sim b$.

(Proof: see [KLM 90].)

7.9 Proposition Let B be a Boolean algebra, and let $(X, \leq, \varphi)$ be a preferential model of B. Then there is a topology on X such that, for all $a, b \in B$,

$a \rightarrow b$ is true in the topological model (X, φ)

(only) if

$\varphi(a) \mid\!\sim_{\leq} \varphi(b)$.

Proof: After a slight modification, the proof of Theorem 7.5 will suffice.

□

7.10 Corollary Let B be a Boolean algebra, and $\mid\!\sim$ a binary relation on B. Then $\mid\!\sim$ is a preferential consequence relation (only) if there is a topological model (X, φ) such that, for all $a, b \in B$,

$a \rightarrow b$ is true in (X, φ) (only) if $a \mid\!\sim b$.

It follows from Proposition 7.9 that the class of preferential models can be seen as a subclass of the class of the topological models. How about the converse? Of course, we have, as a direct corollary of Proposition 7.8 (substituting $\mathcal{P}(X)$ for B and "$\rightarrow$" for "$\mid\!\sim$") :

7.11 Corollary Let X be a topological space. Then there exists a partial ordering $(Y, \leq)$ and a Boolean translation $\varphi : \mathcal{P}(X) \to \mathcal{P}(Y)$ such that

i) for all $a, b \subseteq X$, $a \to b$ (only) if $\varphi(a) \vdash_{\leq} \varphi(b)$,

ii) for all $a \subseteq X$, $\varphi(a)$ is smooth in $(Y, \leq)$.

Hence, the non-monotony involved in, say, the Euclidean plane can, in principle, also be achieved by a Boolean translation from $\mathcal{P}(E)$ into some preferential model. From a geometrical point of view, though, this construction would be highly unnatural. Moreover :

7.12 Proposition There is no partial ordering $(E, \leq)$ on the Euclidean plane such that, for all $a, b \subseteq E$, $a \to b$ (only) if $a \vdash_{\leq} b$.

Proof : For any such partial ordering, and all $p, q \in E$ such that $p \neq q$, $\{p, q\} \to \{p\}$ is not true, hence $\{p, q\} \vdash_{\leq} \{p\}$ is not true, hence $p \leq q$ is not true. Hence, $p \leq q$ (only) if $p = q$, and $a \vdash_{\leq} b$ (only) if $a \subseteq b$.

Hence, $a \to b$ (only) if $a \subseteq b$, which is not true in the Euclidean plane. Hence, there can be no such partial ordering.

□

The same argument applies to most (non-trivial) Hausdorff spaces. Hence, even for a finite Boolean algebra, **the class of topological models is much broader than the class of preferential models.**

On the other hand, we have :

7.13 Proposition For every *finite* topological space X, there is one and only one partial ordering $(X, \leq)$, such that, for all $a, b \subseteq X$,

$$a \to b \text{ (only) if } a \vdash_{\leq} b.$$

Proof : It is easy to see that there is at most one such a partial ordering.

To see that there is at least one, define, for every $p \in X$, O_p to be the smallest open set containing p (which exists because X is finite). Now define the following strict partial ordering on X: (for $p,q \in X$:)

$$p < q \text{ iff } O_p \subset O_q \qquad \text{(strict subset)}$$

and the associated ordinary partial ordering:

$$p \leq q \text{ iff } p < q \text{ or } p = q.$$

Suppose that $a \to b$. Let p be minimal in a. Then there is an O such that

$$O \subseteq O_p,\ O \cap a \neq \varnothing,\ O \cap a \subseteq b.$$

Say $q \in O \cap a$, then $O_q \subseteq O_p$ and $O_q \cap a \subseteq b$.

Since p is minimal in a, $O_q = O_p$, hence $O_p \cap a \subseteq b$, hence $p \in b$.
Hence, $a \rightarrow b$ implies $a \vdash_{\leq} b$.
On the other hand, suppose that $a \vdash_{\leq} b$. For every O such that $O \cap a \neq \emptyset$, choose $p \in a \cap O$. Then, since X is finite, there is a q such that $q \leq p$ and q is minimal in a.
For such q, if $r \in O_q \cap a$ then $O_r \subseteq O_q$, hence $O_r = O_q$ by the minimality (in a) of q. Hence r is minimal in a, hence $r \in b$ (since $a \vdash_{\leq} b$).
Hence, $O_q \cap a \subseteq b$, and $O_q \cap a \neq \emptyset$ (since $q \in O_q \cap a$).
Hence, $a \vdash_{\leq} b$ implies $a \rightarrow b$.
□

Another possible source of confusion is the following :
Corollary 7.6 (7.10), in combination with Definition 7.1, suggests that the open-lemma, the dense-lemma and the closed-lemma (Lemmas 5.17, 5.18 and 5.19, respectively) should be somehow derivable from (the properties expressed by) P1-P5. Although there seems to be no direct derivation of 5.17, 5.18 and 5.19 from P1-P5, they are valid in every topological space and, since the class of topological models contains the class of preferential models, these lemmas state sensible facts about preferential models as well. However, attempts to state those facts without any reference to topology would lead to difficulties. For example, given a partial ordering on a finite space, there is, in general, more than one topology leading to the same conditional. And for each topology, the lemmas state some facts.
Perhaps the situation is best described as follows : although 5.17, 5.18 and 5.19 may seem, at first sight, to go beyond P1-P5/P6, they do not constitute a restriction on the class of models.

§8 Interim evaluation

We have used a topological notion to provide a simple system of non-monotonic reasoning. We also have a characterization by means of a sound and complete axiom system, P1-P5/P6. However, as a mathematical investigation of practical reasoning, this system has its limitations. One should note, in the first place, the lack of expressive power of the language used. To account for, say, the example from §1 (about John not being home at six o'clock), we would like to have a language closed under $\wedge$, $\vee$, $\neg$ and $\rightarrow$, where $\wedge$, $\vee$ and $\neg$ have their usual interpretation, while $\rightarrow$, signifying implication-as-a-rule, has a meaning that forces the inference relation to be non-monotonic, in that b is inferred from "$a \to b$, a", but not from "$a \to b$, a, $\neg b$". However, our language only contains sentences of the form $a \to b$, where a and b contain only $\wedge$, $\vee$, and $\neg$. With our definitions so far, sentences like $(a \to b) \to c$ or $(a \to b) \wedge a \to b$ are meaningless. And in so far as there is talk of an inference relation between defeasible conditionals, it is not a non-monotonic one. Hence, the system of §6 fails to represent a rather essential aspect of defeasible conditionals.

But even within the boundaries of this limited setup, there are two issues to be discussed. The first concerns the suitability of the axiom system P1-P5/P6 for the study of practical reasoning. The second will concern the suitability of Definition 5.7 and of using topology instead of preferential semantics (or some form of probability theory).

To start with the first, the axiom system P1-P5/P6 is, as is commonly acknowledged in the literature on non-monotonic logic, too weak in some instances of practical reasoning, while too strong in others. Very famous is the "irrelevance problem": The fact that $a \mathrel{|\!\sim} b$ does not imply $a \wedge c \mathrel{|\!\sim} b$ enables us to represent situations like "birds can fly", "birds that are penguins cannot fly". But it also prevents inference of "birds that are black can fly" from "birds can fly". This inference, however, seems to be natural, since the property of being black is irrelevant, in that it is unrelated to the animal's ability to fly. Thus, although P1-P5/P6 is weak enough to allow conditionals to be defeasible (that is, to have exceptions), it is too weak to let these defeasible conditionals behave as *rules*-with-possible-exceptions. For, an essential characteristic of such rules in practical reasoning seems to be that *rules are to be applied, unless there is some good reason not to*. For example, in the presence of the rule "any bird can, as a rule, fly", we should conclude "x can fly" from "x is a bird and x is black", as the information "x is black" does not give us any reason not to apply the rule.

In the literature on non-monotonic logic, several strengthenings of P1-P5 have been suggested, typically intended to capture this general principle. Very well-known are the following six candidates (see [Makinson 94]).

(R1) If $a \mathrel{|\!\sim} b$ and $a \wedge c \wedge b \mathrel{|\!\not\sim} \perp$, then $a \wedge c \mathrel{|\!\sim} b$.

This rule amounts to:

If "if a then b" is (a) valid (rule), and $a \wedge c$ is some situation in which it could be applied, then it should be applied, unless the result would be an inconsistent state of knowledge.

This property turns out to be unreasonably strong (see below).

(R2) If $a \mid\sim b$ and $a \wedge c \mid\not\sim \neg b$, then $a \wedge c \mid\sim b$.

This rule amounts to:
If "if a then b" is (a) valid (rule), and $a \wedge c$ is some situation in which it could be applied, then it should be applied, unless $\neg b$ is explicitly concluded from $a \wedge c$.

Although this principle is strictly weaker than R1, it is still unreasonably strong. R2 (as well as R1) claims that the following situation is inconsistent: "birds can fly", "this unknown species, of which we only found some bones, is to be classified as being a (species of) bird, but we do not know whether it could fly or not" (see Example 6.5).

(R3) If $a \mid\sim b$ and $a \mid\not\sim \neg c$, then $a \wedge c \mid\sim b$.

Of all the rules mentioned here, R3 is the best known (see [KLM 90]) and the most commonly accepted as a plausible candidate. However, Example 6.6 shows that the Euclidean plane does not satisfy R3. Besides that, the intuitive story accompanying that example shows that it is questionable whether R3 is valid in practical reasoning.

(R4) If $a \vee b \mid\sim c$ and $a \mid\not\sim c$, then $b \mid\sim c$;
or: If $a \vee b \mid\sim c$, then either $a \mid\sim c$ or $b \mid\sim c$.

If the union of two areas looks green, by and large, then at least one of those areas looks green, by and large, as well. Although R4 seems to have a geometrical motivation, the Euclidean plane does not satisfy R4 (nor R5, below).

(R5) If $a \mid\sim b$ and $a \wedge c \mid\not\sim b$, then $a \wedge \neg c \mid\sim b$;
or: If $a \vee b \mid\sim c$ and $a \wedge b = \bot$ then either $a \mid\sim c$ or $b \mid\sim c$.

Although R5 is much weaker than R3 or R1, example 6.7 shows that there is reason to consider even R5 to be too strong.

And, finally :
(R6) If $a \mid\sim b$ and $a \wedge c \mid\sim \neg b$, then $a \mid\sim \neg c$;

or: If $a \mathrel{|\!\sim} b$ and $a \mathrel{|\!\not\sim} \neg c$, then $a \wedge c \mathrel{|\!\not\sim} \neg b$.

R6 is not a real strengthening of P1-P5/P6, since any preferential consequence relation satifies R6.
(Proof: Using 5.12 iv, 5.10 iii and P2, respectively, we have: if $a \mathrel{|\!\sim} b$ and $a \wedge c \mathrel{|\!\sim} \neg b$, then $a \mathrel{|\!\sim} \neg c \vee \neg b$, hence $a \mathrel{|\!\sim} b \wedge (\neg c \vee \neg b)$, hence $a \mathrel{|\!\sim} \neg c$.)

Assuming P1-P5/P6, each of these properties is (strictly) stronger than its successor. That R1 implies R2 follows from the fact that $a \wedge c \wedge b \mathrel{|\!\sim} \bot$ implies $a \wedge c \mathrel{|\!\sim} \neg b$. To see that R2 implies R3, note that R6 + R2 trivially implies R3. R3 implies R4: if $a \vee b \mathrel{|\!\sim} c$ and $a \mathrel{|\!\not\sim} c$, then $a \vee b \mathrel{|\!\not\sim} \neg b$ (otherwise $a \vee b \mathrel{|\!\sim} a$ would be true, hence $a = (a \vee b) \wedge a \mathrel{|\!\sim} c$), hence, by R3, $b = (a \vee b) \wedge b \mathrel{|\!\sim} c$. That R4 implies R5 is trivial.

Returning to the irrelevance problem, R3 seems to be the principle underlying the inference of "birds that are black can fly" from "birds can fly", if we assume that the sentence "birds are typically non-black" is not true. But R3 is too weak to guarantee, for example, "birds that are green can fly" (assuming that "birds are typically non-green" is true). That inference, though, is not less natural, since the property of being green is equally unrelated to the ability to fly. For other purposes, however, R3 (and even R5) is too strong (see Examples 6.6 and 6.7). Hence, none of the rules R1 to R6 is entirely convincing. In short, the axiom system P1-P5/P6 is too weak to capture the behaviour of a defeasible conditional as a rule-with-possible-exceptions, and this deficit does not have a straightforward solution.
In other respects, though, P1-P5 is too strong. Although the axioms are acceptable as general principles, they do not, of course, constitute hard mathematical properties of defeasible conditionals as occurring in practical reasoning. Some researchers criticize P3, P4 or P5, at least in their full generality. Moreover, in practical reasoning even seemingly uncontroversial statements like "$c \to a \wedge b$ implies $c \to a$" may have exceptions. (In this case, a could be a half truth, only acceptable as a fair account when accompanied by b.) Hence, it seems that in practical reasoning, axioms like P1-P5, but also R1-R6, function themselves, at best, as rules-with-possible-exceptions.

The considerations so far all concerned the system of §6, a system that uses Definition 5.7 in a rather simplistic way. In the rest of this thesis we will

hold on to Definition 5.7, but we will use it in more sophisticated ways, taking the above considerations into account, as much as possible. There is a number of reasons to maintain Definition 5.7 as the central subject of study. In part, these arguments cannot be given until after Chapter 3 or Chapter 5. On the other hand, some of the arguments have already been given in §7. In comparison to preferential semantics, our topological presentation enables us to state and prove insights that would otherwise be hard to achieve (for example, the Lemmas 5.17, 5.18, 5.19, or Theorems 8.1, 8.3 and 8.7 below). Moreover, as said in §7, the class of topological models is strictly larger than the class of preferential models. However, our most important argument, at this moment, is the psychological advantage over preferential semantics.
Judging from 5.5 v), the definition itself is less "alien" to ordinary mathematics ("nowhere dense subset" is a standard notion from topology) than definitions using preferential (or probabilistic) ideas. But especially the intuitive content of Definition 5.7 depicts defeasible conditionals as an ordinary notion, useful in modelling rational human behaviour, rather than as a twisted notion, intended to model sloppy or lazy reasoning.

What remains to be done, for the moment, is a further reflection on Definition 5.7 itself. In order to defend Definition 5.3 (thus, indirectly, also Definition 5.7), let us discuss some useful facts and theorems, first.

If X is a topological space, then, according to the dense lemma (Proposition 5.18), $a \to b$ implies $a \cap c \to b$ if c is dense in a. Since $a \not\to c^c$ is true (only) if c is somewhere dense in a, R3 amounts to

" $a \to b$ implies $a \cap c \to b$ if c is somewhere dense in a ",

while every topological space satisfies

" $a \to b$ implies $a \cap c \to b$ if c is (everywhere) dense in a ".

The motivation behind R3 seems to be that "$a \not\to c^c$" expresses that "x $\in c$" is not a ridiculous assumption relative to "x $\in a$". In view of this, it is good to know the following:

8.1 Theorem Let X be a topological space X and $a, b \subseteq$ X. Then

$a \to b$ (only) if for every c that is dense in a, b is dense in $a \cap c$.

The proof of this theorem will be given after Corollary 8.5, below.

Inspired by R3, let us read "b is not a ridiculous assumption relative to a" instead of "b is everywhere dense in a". (For example, "x is a penguin" is a ridiculous assumption relative to "x is a bird".) Then Theorem 8.1 reads:
b is plausibly concluded from a (only) if there is no non-ridiculous extra assumption c, relative to which b sounds ridiculous or, alternatively, (only) if b sounds plausible in all ordinary (= non-ridiculous relative to a) situations $a \wedge c$.

Closely related is the following alternative characterization of full subsets. Let X be a topological space. $\mathcal{F} := \{ a \subseteq X \mid X \to a \}$, the collection of all full subsets of X.

8.2 Definition $\mathcal{V} \subseteq \mathcal{P}(X)$ is called a *dense-filter* (on X) if it is a filter of the Boolean algebra $(\mathcal{P}(X), \subseteq)$ (that is: for all a, $b \in \mathcal{V}$, $a \cap b \in \mathcal{V}$, and for all $a, b \subseteq X$ such that $a \in \mathcal{V}$ and $a \subseteq b$: $b \in \mathcal{V}$) and

for all $a \in \mathcal{V}$, a is dense in X.

$\mathcal{V}$ is called a *maximal dense-filter* if it is a dense-filter, and for every dense-filter $\mathcal{V}'$ such that $\mathcal{V} \subseteq \mathcal{V}'$: $\mathcal{V} = \mathcal{V}'$.

(A maximal dense-filter is a dense-filter which is maximal as a dense-filter. That is not the same thing as a maximal filter that is also a dense-filter. Note that $\mathcal{P}(X)$ is not a dense-filter, unless $X = \varnothing$.)

8.3 Theorem $\mathcal{F}$ is the intersection of all maximal dense-filters.

To prove both Theorem 8.1 and 8.3, we will need the following two results.

8.4 Lemma (d-lemma)

If $a \subseteq X$ is not full in X, then there is a $d \subseteq X$ such that

d is dense in X,

$d \cap a$ is not dense in X.

Proof: Suppose that a is not full in X. Then a^c is somewhere dense in X, say $O \neq \varnothing$, a^c is dense in O. Define d to be $O^c \cup a^c = (O \cap a)^c$.

Then $d \cap a \cap O = \varnothing$, $O \neq \varnothing$, hence $a \cap d$ is not dense in X.

For every nonempty O', either $O' \cap O^c \neq \varnothing$, or $O' \subseteq O$.

If $O' \cap O^c \neq \varnothing$, then $O' \cap d \neq \varnothing$.

If $O' \subseteq O$, then $O' \cap a^c \neq \varnothing$ (a^c is dense in O), hence $O' \cap d \neq \varnothing$.

Hence, d is dense in X.

□

8.5 Corollary

If $a \subseteq X$ is not full in X, then there is a $d \subseteq X$ such that

d is dense in X,

a is not dense in d (that is, $d \cap a$ is not dense in d).

Proof: Define d as in the proof of 8.4. Then $d \cap a$ is not dense in d, since $O \cap d \neq \varnothing$ (since $O \cap a^c \neq \varnothing$), while $O \cap d \cap a = \varnothing$.

□

Proof of Theorem 8.3:

Suppose that $a \in \mathcal{J}$ and $\mathcal{V}$ is a maximal dense-filter. Then

$a + \mathcal{V} := \{ a' \cap v \mid a \subseteq a', v \in \mathcal{V} \}$ is a dense-filter,

$\mathcal{V} \subseteq a + \mathcal{V}$ and $a \in a + \mathcal{V}$.

But $\mathcal{V}$ is maximal, hence $\mathcal{V} = a + \mathcal{V}$ and $a \in \mathcal{V}$.

(As said, $\mathcal{P}(X)$ is not a dense-filter, unless $X = \varnothing$.)

Hence, a is an element of every maximal dense-filter.

On the other hand, suppose that $a \notin \mathcal{J}$.

Then by 8.4 there is a dense set d such that $a \cap d$ is not dense in X.

Then $\{ d' \subseteq X \mid d \subseteq d' \}$ is a dense-filter.

Let $\mathcal{V}$ be a maximal dense-filter containing this collection.

Then $a \notin \mathcal{V}$. (To see this, note that $a \in \mathcal{V}$ would imply $a \cap d \in \mathcal{V}$.
Note that dense-filters (like $\mathcal{V}$) contain only dense subsets of X and that $a \cap d$ is not dense in X.)

Hence, there exists a maximal dense-filter not containing a, which completes the proof.

□

Proof of Theorem 8.1:

Suppose that $a \to b$ and c is dense in a. Then, by the dense lemma,

$a \cap c \to b$, hence b is dense in $a \cap c$.

On the other hand, suppose that $a \not\to b$.

Then in a-with-induced-topology, $a \cap b$ is not a full subset.

By Corollary 8.5, there is a $d \subseteq a$ such that

d is dense in a, and $a \cap b$ is not dense in d

(that is, $d \cap (a \cap b)$ is not dense in d).

Then $d \cap b = d \cap (a \cap b)$ is not dense in d,

hence b is not dense in d.

Hence,

d is dense in a,
and b is not dense in $d = a \cap d$,
which completes the proof.
□

Theorems 8.1 shows that something of the idea behind R3 is present in every topological (or preferential) model. Theorem 8.3 is a rather abstract variant of 8.1. Both might serve as an argument in favour of the Definition 5.3 as a plausible interpretation of inclusion-up-to-possible-exceptions. These theorems are not the most important arguments, however, to appreciate the definition of a full subset.

The notion of a full subset is a natural adaptation of the notion of a dense subset, provided we take points that we draw (rough spots) more seriously than idealizations of such points (say, elements of E). Although it is questionable whether topology, as practised in contemporary mathematics as well as in this thesis, is fit to capture "rough spots", we could consider rough spots as representable by some particular kind of nonempty open sets (for example, circular discs). A better treatment is to be found in [Hjelmslev 23].
A dense subset is a set that is present in every region (that is, a set that contains elements in every nonempty open subset). The geometrical intuition behind this notion is that a dense subset is omnipresent in the topological space. Colouring the points of a dense set blue should make the whole topological space appear blue. But if we take seriously the idea that a single point (infinitely small) of, say, the Euclidean plane is a negligible object, than the definition of a dense subset needs adaptation. Instead of containing a single point in every region, every region should contain a rough point *convincingly contained in the set* (whatever that may mean), before we might consider that set "omnipresent". Adapting the notion of a dense subset along these lines yields our Definition 5.3.
Reconsidering the meaning of "convincingly contained in the set", we might just as well accept the following definition:

8.6 Definition A set $a \subseteq$ X is called *full-2* (in X) whenever every nonempty O contains a nonempty O' such that O' $\rightarrow a$.

This definition is entirely acceptable as an alternative for Definition 5.3. In view of this it is good to know that these two definitions are equivalent :

8.7 Proposition A subset of X is full-2 in X (only) if it is full in X.

Proof: Suppose that $a \subseteq X$ is full. Then every nonempty O contains a nonempty O' such that $O' \subseteq a$, hence $O' \rightarrow a$.

Hence, a is full-2 in X.

On the other hand, suppose that $a \subseteq X$ is full-2. Then every nonempty O contains a nonempty O' such that $O' \rightarrow a$, which, on its turn, contains a nonempty O'' such that $O'' \subseteq a$.

Hence, a is full in X.

□

Hence, the notion of a full subset has the following properties:

i) if a is full in X, then a is dense in X,

ii) if a is full in X and $a \subseteq a' \subseteq X$, then a' is full in X,

iii) if a and b are full in X, then $a \cap b$ is full in X,

iv) a is full in X (only) if every nonempty O contains a nonempty O' such that a is full in O'.

Most probably, however, the notion of a full subset is not the only notion having these properties.

Recapitulation

§5 introduced a topological notion that will be the central notion of this thesis. Several properties of this notion were established. §6 then presented a simple way to use this notion to study practical reasoning. (In §8, this system turned out to be too simple.) In §7, it was proven that the list of properties found in §5 cannot essentially be extended and the system of §6 was compared to the approach of [KLM 90].

Despite the fact that we could use the mathematics of their preferential models to prove our completeness theorem for topological models, the intuitive content of our topological conditional is not compatible with preferential semantics. Hence, for the rest of this thesis, *the reader is urged to forget about any association of "$\rightarrow$" or "$\mid\!\sim$" with the idea of preference of some possible worlds over others.* Likewise, these symbols are not to be associated with probabilistic ideas. Our topological treatment generalizes intuitions of old-fashioned mathematics and depicts defeasible implication as a natural phenomenon that does not need additional associations.

Chapter 3
Non-monotonic Inference

The topological notions of Chapter 2 will be used to give a mathematical caricature of the behaviour of an imaginary person handling the rules of classical propositional (monotonic) logic as if they were rules-with-possible-exceptions. This will be done by providing an interpretation of nested implicational statements. That is, of sentences like $(\mathtt{a} \to \mathtt{b} \wedge \mathtt{c}) \to (\mathtt{a} \to \mathtt{b})$.

§9 Nested Implicational Statements

In all of Chapter 3, L denotes a language generated via $\wedge$, $\vee$, $\to$ and $\neg$, from a finite number of basic formulas, "$\to$" being an additional binary connective. We will use a, b, c, ... to denote subsets of some topological space and a, b, c, ... as variables ranging over L. $\mathtt{a}$, $\mathtt{b}$, $\mathtt{c}$, ... denote basic formulas. Hence, a $\to$ b denotes a formula in L, while $a \to b$ is (not a formula, but) a sentence about two sets, a and b (see Definition 5.7). This multiple use of the symbol "$\to$" should not cause much confusion.

Suppose that X is a set of possible worlds, and suppose that a, b and c are sentences interpreted in all the worlds of X. I.e., for each of these sentences and each world of X it has been determined whether that sentence is true or false in that world. To interpret a sentence like $(\mathtt{a} \to \mathtt{b}) \to \mathtt{c}$, we need to define, for every world in X, what it means for $(\mathtt{a} \to \mathtt{b})$ to be true in that world (rather than in the totality of all the worlds). We will assume, in this chapter, that the extension of $(\mathtt{a} \to \mathtt{b})$ is determined only by the extensions of $\mathtt{a}$ and $\mathtt{b}$: whether $(\mathtt{a} \to \mathtt{b})$ is true or false in a world does only depend on the positions in X of the worlds satisfying $\mathtt{a}$ and $\mathtt{b}$, respectively. In principle, any pair of subsets of X could occur as the extension of a pair of formulas. Let us, therefore, define :

9.1 Definition An *extended model* is a tuple (X, I, φ), where

X is a topological space,

I(,) is a binary operation on $\mathcal{P}$(X), such that

for all $a, b \subseteq$ X, I(a, b) $\subseteq$ X,

φ is a map $L \to \mathcal{P}(X)$ such that, for all a, b $\in$ L,

$$\varphi(a \wedge b) = \varphi(a) \cap \varphi(b),$$
$$\varphi(a \vee b) = \varphi(a) \cup \varphi(b),$$
$$\varphi(\neg a) = \varphi(a)^c,$$
$$\varphi(a \to b) = I(\varphi(a), \varphi(b)).$$

When there is no danger of confusion, we will freely speak about "model X", "model (X, φ)", "model X based on I" etc., instead of "model (X, I, φ)".

9.2 Definition A formula a $\in$ L is called *true in* (X, I, φ), whenever

$$\varphi(a) = X.$$

It will be clear that (X, I, φ) is not for all I(,) acceptable as something that provides sensible interpretations for formulas containing implication. For example, for every a, b $\in$ L, we would like a $\to$ b to be true in (X, I, φ) (only) if $\varphi(a) \to \varphi(b)$ is true in the topological space X.

9.3 Definition I(,) is called an *implication operator* on X whenever

for all $a, b \subseteq X$, $I(a, b) = X$ (only) if $a \to b$.

9.4 Proposition If (X, I, φ) is an extended model, and I(,) is an implication operator on X, then, for every a, b $\in$ L, a $\to$ b is true in (X, I, φ) (only) if $\varphi(a) \to \varphi(b)$ is true in the topological space X.
(Provable by elementary check.)

9.5 Example For every topological space, the "trivial" operator, defined by $T(a, b) = X$ if $a \to b$, and $T(a, b) = \varnothing$ if $a \not\to b$, is an implication operator. The operator Q(,) ("material implication"), defined by $Q(a, b) = a^c \cup b$ is not an implication operator, unless the space is monotonic (see Definition 5.13).

In connection with the geometrical motivation of Definition 5.7, there is a very natural implication operator on every topological space :

9.6 Definition Let X be a topological space, and $a, b \subseteq X$. Then $O(a, b)$ is the union of all (open) $O \subseteq X$ such that $O \cap a \to b$.

Hence, for all w $\in$ X, w $\in O(a, b)$ (only) if there is some O such that w $\in$ O and $O \cap a \to b$. This amounts to the following: a sentence a $\to$ b is

defined to be true in world w whenever "in some neighbourhood of w, practically all worlds that satisfy a do also satisfy b".
Note that $O(a, b)$ is open for all $a, b \subseteq X$.

9.7 Example If E denotes the Euclidean plane, and ℓ a line in E, then $O(E, \ell^c) = E$, $O(\ell, \ell^c) = \ell^c$. Note that $O(E, \ell^c) \cap E \not\subseteq \ell^c$, but $O(E, \ell^c) \cap E \to \ell^c$ is true.

9.8 Proposition Let X be a topological space. Then:
i) for all $a, b \subseteq X$, $O(a, b) \cap a \to b$,
ii) $O(\ , \ )$ is an implication operator.
Proof: Suppose that $O \cap (O(a, b) \cap a) \neq \varnothing$. Say $p \in O \cap (O(a, b) \cap a)$. Then $p \in O(a, b)$, hence, there is an O' such that $O' \cap a \to b$ and $p \in O'$, hence $O \cap (O' \cap a) \neq \varnothing$. Since $O' \cap a \to b$, there is an $O'' \subseteq O$ such that

$$O'' \cap (O' \cap a) \neq \varnothing \text{ and } O'' \cap (O' \cap a) \subseteq b.$$

But $O''' := O'' \cap O' \subseteq O(a, b)$, hence

$$O''' \cap (O(a, b) \cap a) \neq \varnothing \text{ and } O''' \cap (O(a, b) \cap a) \subseteq b.$$

Hence, $O(a, b) \cap a \to b$.
To see that $O(\ , \ )$ is an implication operator, suppose that $a \to b$. Then $X \cap a \to b$, and X is an open set. Hence $O(a, b) = X$. On the other hand, suppose that $O(a, b) = X$. Then i) implies $a \to b$.
□

9.9 Corollary $O(a, b)$ is the largest (open) O such that $O \cap a \to b$. Note that, for all $a, b \subseteq X$, $a^c \cup b$ is the largest $V \subseteq X$ such that $V \cap a \subseteq b$.

9.10 Proposition Let X be a topological space. Then the following statements are equivalent:
i) X has discrete topology (i.e. every subset of X is open),
ii) for all $a, b \subseteq X$, $O(a, b) = a^c \cup b$,
iii) for all $a \subseteq X$, $O(a, \varnothing) = a^c$.
Proof:
i) $\Rightarrow$ ii): If X has discrete topology, then $a \to b$ (only) if $a \subseteq b$, by which ii) easily follows.
ii) $\Rightarrow$ iii) is trivial.
iii) $\Rightarrow$ i): If X satisfies iii), then for all $a \subseteq X$, $a = O(a^c, \varnothing)$, which is an open set.
□

Using the implication operator O(,), we can now interpret rules of inference as follows.

9.11 Definition If $a_1, \dots, a_m, b_1, \dots, b_n$ are formulas of L, then

the *rule* $\dfrac{a_1, \dots, a_m}{b_1, \dots, b_n}$ is called *topologically valid* or *O-valid* whenever the formula $a_1 \wedge \dots \wedge a_m \rightarrow b_1 \wedge \dots \wedge b_n$ is true in every extended model based on O(,) (that is, in every extended model (X, O(,), φ)).

Note that this definition involves nested implicational statements whenever any of the formulas $a_1, \dots, a_m, b_1, \dots, b_n$ contains the symbol "$\rightarrow$".

9.12 Example The rule $\dfrac{a \rightarrow b,\ a}{b}$ is topologically valid (this is an easy corollary of Proposition 9.8 i). However, the rule $\dfrac{a \rightarrow b,\ a,\ \neg b}{b}$ is not topologically valid : if E is the Euclidean plane, and ℓ is a line in E, then $O(E, \ell^c) = E$, and $O(E, \ell^c) \cap E \cap \ell \rightarrow \ell^c$ is not true.

9.13 Theorem $\dfrac{a \rightarrow b}{a \wedge c \rightarrow b}$ is topologically valid.

Proof: Let X be a topological space, and $a, b, c \subseteq X$.
Suppose that $O \cap O(a, b) \neq \varnothing$. We may assume that $O \subseteq O(a, b)$.
Then $O \cap a \rightarrow b$.
If c is dense in $O \cap a$, then $O \cap a \cap c \rightarrow b$, hence $O \subseteq O(a \cap c, b)$.
If c is not dense in $O \cap a$, then there is an $O' \subseteq O$ such that

$$O' \cap a \neq \varnothing, \text{ and } O' \cap a \cap c = \varnothing.$$

Then $O' \cap a \cap c \rightarrow b$, hence $O' \subseteq O(a \cap c, b)$.
Moreover, $O' \cap O(a, b) \neq \varnothing$, since $O' \neq \varnothing$.

In both cases, there exists an $O' \subseteq O$ such that

$$O' \cap O(a, b) \neq \varnothing \text{ and } O' \subseteq O(a \cap c, b).$$

□

9.14 Corollary If X is a topological space, and $a, b \subseteq X$, such that $a \rightarrow b$, then, for all $c \subseteq X$, there is an open subset V of X such that

$X \rightarrow V$ (practically all worlds are in V)

$V \cap a \cap c \to b$ (under the extra assumption "$w \in V$", $w \in a \cap c$ implies, as a rule, $w \in b$.)

(Namely $V = O(a \cap c, b)$.)

A special case of Theorem 9.13 is the topological validity of $\dfrac{a \to b}{a \wedge \neg b \to b}$.
In this special case, the above remarks yield the existence of a set V such that $X \to V$ and $V \cap a \cap b^c \to b$. However, in this special case, it is clear that we could take $V = a^c \cup b$, since $a \to b$ implies $X \to a^c \cup b$ and $(a^c \cup b) \cap a \cap b^c \to b$. Hence, we could say that this special case of Theorem 9.13 is nothing more than the following triviality: "if $a \to b$ is true, then the set of its counterexamples (that is, $a \cap b^c$) is a negligible minority in X."

Warning Theorem 9.13 allows us to think of the rule of monotony as *valid*, be it only up to possible exceptions. Our system is non-monotonic, nevertheless, in that "$a \to b$ is true in the extended model (X, O, φ)"
does not imply "$a \wedge c \to b$ is true in the extended model (X, O, φ)".
Nor does "$a \to b$ is true in every extended model (X, O, φ)"
imply "$a \wedge c \to b$ is true in every extended model (X, O, φ)".
In other words, the topological validity of $\dfrac{a}{b}$ does not imply the topologically validity of $\dfrac{a \wedge c}{b}$ (see Example 9.12).
In general, *the topological validity of* $\dfrac{a}{b}$ and $\dfrac{b}{c}$ *does not imply the topologically validity of* $\dfrac{a}{c}$.

9.15 Theorem Each of the following rules is topologically valid:

$$\frac{c \to a,\ c \to b}{c \to a \wedge b} \qquad \frac{c \to a \wedge b}{c \to a} \quad \frac{c \to a \wedge b}{c \to b}$$

$$\frac{c \to a}{c \to a \vee b} \quad \frac{c \to b}{c \to a \vee b} \qquad \frac{c \to a \vee b,\ c \wedge a \to d,\ c \wedge b \to d}{c \to d}$$

$$\frac{c \wedge a \to b}{c \to \neg a \vee b} \qquad \frac{c \to a,\ c \to \neg a \vee b}{c \to b}$$

$$\frac{c \wedge a \rightarrow b,\ c \wedge a \rightarrow \neg b}{c \rightarrow \neg a} \qquad \frac{c \rightarrow a,\ c \rightarrow \neg a}{c \rightarrow b}$$

As well as:

$$\frac{}{a \rightarrow a} \qquad \frac{c \wedge a \wedge b \rightarrow d}{c \wedge b \wedge a \rightarrow d} \qquad \frac{c \wedge a \wedge a \rightarrow b}{c \wedge a \rightarrow b}$$

$$\frac{a \rightarrow b,\ a}{b} \ \text{(modus ponens)} \qquad \frac{c \rightarrow a}{c \wedge b \rightarrow a} \ \text{(monotony)}$$

Proof: Let X be a topological space, and $a, b, c, d \subseteq$ X.

For most of the rules above, it is possible to prove an even stronger statement, as follows.

If $O_1 = O(c, a)$ and $O_2 = O(c, b)$ then $O_1 \cap O_2 \cap c \rightarrow a$ and $O_1 \cap O_2 \cap c \rightarrow b$ (by the open-lemma).

Hence $O_1 \cap O_2 \cap c \rightarrow a \cap b$, hence $O_1 \cap O_2 \subseteq O(c, a \cap b)$.

Hence, $O(c, a) \cap O(c, b) \subseteq O(c, a \cap b)$, which immediately implies

$O(c, a) \cap O(c, b) \rightarrow O(c, a \cap b)$.

Likewise :

$O(c, a \cap b) \subseteq O(c, a)$, $O(c, a \cap b) \subseteq O(c, b)$,

$O(c, a) \subseteq O(c, a \cup b)$, $O(c, b) \subseteq O(c, a \cup b)$,

$O(c, a \cup b) \cap O(c \cap a, d) \cap O(c \cap b, d) \subseteq O(c, d)$,

(since $(O \cap c) \cap a \rightarrow d$, $(O \cap c) \cap b \rightarrow d$ implies

$(O \cap c) \cap (a \cup b) \rightarrow d$

and $(O \cap c) \cap (a \cup b) \rightarrow d$, $(O \cap c) \rightarrow a \cup b$ implies

$(O \cap c) \rightarrow d$)

$O(c \cap a, b) \subseteq O(c, a^c \cup b)$,

(since $v \cap a \rightarrow b$ implies $v \rightarrow a^c \cup b$)

$O(c, a^c \cup b) \cap O(c, a) \subseteq O(c, b)$,

(since $(a^c \cup b) \cap a \subseteq b$)

$O(c \cap a, b) \cap O(c \cap a, b^c) \subseteq O(c, a^c)$,

$O(c, a) \cap O(c, a^c) \subseteq O(c, b)$.

The topological validity of modus ponens and the rule of monotony was established in Proposition 9.8 i) and Theorem 9.13, respectively. The rest is trivial.

□

The importance of this theorem will be explained in the next section.

§10 Defeasible Rules of Inference

Let us think of an imaginary person who uses the rules of propositional logic as rules-with-possible-exceptions. He knows about the distinction between object-language and meta-language. On the object level, he continuously uses $\neg ... \vee ...$ to interpret implicational statements. On the metalevel, he draws conclusions using the rules of inference of Theorem 9.15. The symbol "$\to$" is the symbol that *we* use to denote the person's deducibility relation (it is not part of the person's object-language). For example, what we write as

$$\frac{a \to b, \quad a \to c}{a \to b \wedge c}$$

is known by him as "if b is deducible from a, and c is deducible from a, then b-and-c is deducible from a" or, simpler, as the process of writing

$$\frac{b \qquad c}{b \wedge c}$$

somewhere within an argument or proof tree. (Note that modus ponens, in the form by which it was included in Theorem 9.15, also functions as such a rule in practical reasoning: if a and b are formulas such that b is deducible from a, and, in some situation, with some concrete interpretation of the basic formulas occurring in a or b, a is true, then modus ponens allows us to conclude that b, with the same interpretation of the basic formulas, is also true in that situation.)

However, our imaginary person handles some of these rules (namely modus ponens and the rule of monotony) as if they were rules-with-possible-exceptions. The notion of topological validity, now, can be seen as a mathematical image of the reasoning behaviour of such a person. Thus, our imaginary person appreciates each of the laws of classical propositional logic, distinguishes between implication ($\neg ... \vee ...$) and deducibility ($\to$), uses an unambiguously defined and fixed language (having $\wedge$, $\vee$, and $\neg$), distinguishes between an objectlanguage and a metalanguage etc, etc. Note that the terms objectlanguage and metalanguage are meant, here, to refer to the person's objectlanguage and metalanguage. They are not to be confused with *our* objectlanguage and metalanguage, when reading, for example, Definition 9.11 or Theorem 9.15. For example, "$\to$" is not part of the person's object language.

The rules of inference of Theorem 9.15, when interpreted as valid-without-exceptions, are known to be a complete characterization of classical propositional logic. Thus, the person could be said to reason non-monotonically "on the metalevel" while accepting each of the laws of classical propositional (monotonic) logic as valid-with-possible-exceptions.

The laws that the person appreciates as valid (be it only up-to-possible-exceptions) include the principle of monotony: the person appreciates "if c is deducible from the assumptions $a_1, \dots, a_n$, then c is deducible from $a_1, \dots, a_n, a_{n+1}$" as valid. Nevertheless, "if c is deducible from the assumptions $a_1, \dots, a_n$, and c is not deducible from $a_1, \dots, a_n, a_{n+1}$" is regarded as a consistent situation :

$$\frac{a_1 \wedge \cdots \wedge a_n \to c}{a_1 \wedge \cdots \wedge a_n \wedge a_{n+1} \to c} \quad \text{is topologically valid, but}$$

$$\frac{a_1 \wedge \cdots \wedge a_n \to c, \quad \neg(a_1 \wedge \cdots \wedge a_n \wedge a_{n+1} \to c)}{a_1 \wedge \cdots \wedge a_n \wedge a_{n+1} \to c} \quad \text{is not.}$$

Likewise, it follows by Example 9.12 that modus ponens, as handled by our person, is accepted as valid, but may have exceptions. It is important to note that our person handles the other rules of Theorem 9.15 as rules *without* exceptions. This is easily seen by reconsidering the proof of Theorem 9.15.

For example, $\dfrac{a \to b, \quad a \to c, \quad d}{a \to b \wedge c}$ is topologically valid.

(For all X and all $a, b, c \subseteq X$, $O(c, a) \cap O(c, b) \subseteq O(c, a \cap b)$.)

Hence, if our person has concluded that b-and-c is deducible from a on the grounds that both b and c are deducible from a, this conclusion will not be withdrawn on the arrival of whatever new information (d).

Hence, although the mathematical elaboration may leave room for improvement, Theorem 9.15 can be said to support the following tentative conclusion. The typical monotonic character of classical propositional logic is not a consequence of any of the following :

1) accepting the rule of monotony,
2) accepting that rule and all "classical laws",
3) using an unambiguously defined formal language,

4) making a distinction between objectlanguage and metalanguage, or between implication and inference,
5) using $\neg\ldots\vee\ldots$ to interpret implication on the objectlevel,
6) any combination of 1) - 5).

It is a consequence of *adopting monotonic reasoning habits on the metalevel.* The collection of axioms that is usually assumed to characterize propositional logic does not suffice to characterize these reasoning habits. Most probably, there is no other set of rules that manages to capture, completely, these habits, since any such set of rules will involve implicational statements, in one way or another, that will be susceptible of a non-strict interpretation. This latter statement, however, is no more than a conjecture, and is certainly not supported by Theorem 9.15. Let us state it, more provocatively, as follows : in Chapter 1, it was defended as thinkable that non-monotonic reasoning defies complete axiomatization or formalization. Likewise, however, it is thinkable that monotonic reasoning defies complete axiomatization in the eyes of a person that persistently reasons non-monotonically.

There is a number of results that reinforce the above interpretation of Theorem 9.15. The first we will present is Theorem 10.3, below. This result states that there is no difference between topological validity and classical validity, provided we restrict ourselves to rules of a very simple kind, thus generalizing Theorem 9.15. Other, more subtle, reinforcements will be an "alternative definition" of the notion of topological validity (Definition 12.11), Boutilier's Theorem (Theorem 12.24), and Corollary 12.25.

10.1 Definition A formula (in L) is called *classical* whenever it does not contain the symbol "$\rightarrow$". If a and b are classical formulas, then $a \rightarrow b$ is called a *standard* formula. (Other formulas are not.)

A rule $\dfrac{A_1, \ldots, A_m}{B_1, \ldots, B_n}$ is called a *standard rule* whenever each one of $A_1, \ldots, A_m, B_1, \ldots, B_n$ is a standard formula.

10.2 Definition A *classical model* is an extended model based on material implication, that is, an extended model (X, Q , φ) where $Q(a, b) = a^c \cup b$ for all $a, b \subseteq X$.

A rule $\dfrac{A_1, \dots, A_m}{B_1, \dots, B_n}$ is called *classically valid* whenever the formula

$A_1 \wedge \dots \wedge A_m \rightarrow B_1 \wedge \dots \wedge B_n$ is true in every classical model.

As is well kown, if $a_0, \dots, a_n, b_0, \dots, b_n$ are classical formulae, then the rule

$$\frac{a_1 \rightarrow b_1, \dots, a_n \rightarrow b_n}{a_0 \rightarrow b_0}$$

is classically valid (only) if for every collection, M, of possible worlds in which $a_1 \vDash_M b_1, \dots, a_n \vDash_M b_n$ are true, $a_0 \vDash_M b_0$ is also true.

10.3 Theorem A standard rule $\dfrac{A_1, \dots, A_m}{B_1, \dots, B_n}$ is topologically valid (only) if it is classically valid.

(For a related result, see Theorem 11.12.)

Proof: If the rule is topologically valid, it is valid in all models with discrete topology. But in those models, $O(a, b) = a^c \cup b$ for all subsets a, b (Proposition 9.10). Hence the rule is classically valid.

On the other hand, if the rule is classically valid, it can be proved using the rules from Theorem 9.15. Now it suffices to notice (supposing all appearing rules to be standard) :

i) If $\dfrac{A_1, \dots, A_m}{B_1, \dots, B_n}$ is topologically valid (and C is a standard formula), then $\dfrac{A_1, \dots, A_m, C}{B_1, \dots, B_n}$ is topologically valid.

(Proof: For all X and all $a, b \subseteq X$, $a \rightarrow b$ implies $a \cap O \rightarrow b$ for open O, and every standard formula is, in every extended model, interpreted by an open set, since $O(a, b)$ is open for all $a, b \subseteq X$, and all X.)

ii) If $\dfrac{A_1, \dots, A_m}{B_1, \dots, B_n}$ and $\dfrac{B_1, \dots, B_n}{C_1, \dots, C_k}$ are topologically valid, then

$$\frac{A_1, \dots, A_m}{C_1, \dots, C_k} \text{ is also topologically valid.}$$

(Proof: Use i) above and Proposition 5.11 ii).)

□

For example, $\dfrac{a \to b}{\neg b \to \neg a}$ is topologically valid, since,

for all X and all $a, b \subseteq X$,

$$\begin{aligned}
&O(a, b) \to O(a \cap b^c, b),\\
&O(a \cap b^c, b) \cap O(a \cap b^c, b^c) \subseteq O(a \cap b^c, a^c),\\
&O(a \cap b^c, a^c) \cap O(a^c \cap b^c, a^c) \subseteq O(b^c, a^c), \text{ and}\\
&O(a \cap b^c, b^c) = O(a^c \cap b^c, a^c) = X.
\end{aligned}$$

The core of this proof seems to be aptly represented by the following scheme:

$$\dfrac{\dfrac{\dfrac{a \to b}{a \wedge \neg b \to b} \quad a \wedge \neg b \to \neg b}{a \wedge \neg b \to \neg a} \qquad \neg a \wedge \neg b \to \neg a}{\neg b \to \neg a}$$

Warning In the sequel, we will give some of the proofs by just showing such a scheme, representing only the core of the proof. Typically, these schemes can easily be turned into complete proofs by repeated use of the open-lemma, Theorem 9.13, validity of rules previously achieved, etc. However, the schemes in itself do typically not constitute complete proofs.

§11 Topologically Valid Rules

Let us examine topologically valid rules more closely. Theorem 10.3 tells us which standard rules are topologically valid. Before we study the topological validity of non-standard rules, let us first make some rather obvious remarks.

11.1 Definition A substitution is a function $\sigma: L \to L$ such that,

for all $a, b \in L$,

$$\begin{aligned}
&\sigma(a \wedge b) = \sigma(a) \wedge \sigma(b),\\
&\sigma(a \vee b) = \sigma(a) \vee \sigma(b),\\
&\sigma(\neg a) = \neg \sigma(a),\\
&\sigma(a \to b) = \sigma(a) \to \sigma(b).
\end{aligned}$$

If $a_1, \ldots, a_n$ are the basic formulas of L, then for every row $A_1, \ldots, A_n$ of formulas of L there is one and only one substition σ such that $\sigma(a_i) = A_i$, for $i = 1, \ldots, n$. On the other hand, for every substition there is one and only one row $A_1, \ldots, A_n$ satisfying those conditions.

11.2 Theorem (Substitution theorem)
If $A_1, \ldots, A_m, B_1, \ldots, B_n$ are formulas of L, such that

$$\frac{A_1, \ldots, A_m}{B_1, \ldots, B_n}$$

is topologically valid, then for every substitution σ,

$$\frac{\sigma(A_1), \ldots, \sigma(A_m)}{\sigma(B_1), \ldots, \sigma(B_n)}$$

is also topologically valid.
Proof: This is a trivial consequence of the definition of topological validity (Definition 9.11).

It is not straightforward to explain the "sense" of non-standard rules. This is not because non-standard rules do not make sense, but because there is a variety of possible views. Of course, the notion of topological validity does not depend on the choice of a particular point of view. Therefore, we will not exhaustively distinguish and explain all kinds of viewpoints.
One possibility is, to extend the story of our imaginary person of §10. Let us, therefore, suppose that this imaginary person introduces the symbol "$\rightarrow$" in his object language to denote implication-with-possible-exceptions. Let us also assume that the person appreciates precisely the topologically valid rules as valid. We will use two different symbols, "$\vdash$" and "$\rightarrow$", corresponding to the person's distinction between deducibility and implication. *We*, on the other hand, treat these two symbols as synonyms. For example :

11.3 Example (Concerning modus ponens)

i) $$\frac{a \vdash b}{c \vdash a \rightarrow b}$$

is topologically valid, since, for every topological space X, and all $a, b \subseteq X$, $O(a, b) \cap c \rightarrow O(a, b)$, hence $O(a, b) \subseteq O(c, O(a, b))$, hence
$O(a, b) \rightarrow O(c, O(a, b))$.

ii) $\dfrac{c \vdash p, \quad c \vdash p \to q}{c \vdash q}$ is topologically valid.

(Proof: This and some of the following examples will be an easy corollary of Theorem 11.5 below.)

ii) states that, if our person observes that a certain context supports the proposition p, as well as p $\to$ q, then q is concluded to be supported by that context. In other words : the person appreciates

$$\frac{p, \quad p \to q}{q}$$

as a valid proof-step.
On the other hand,

iii) $\dfrac{c \vdash p, \quad c \vdash p \to q, \quad c \nvdash q}{c \vdash q}$ is not topologically valid.

(Proof: In the Euclidean plane E, let c denote a line, $p = \mathrm{E}$ and $q = c^{\mathrm{c}}$. Then $\mathrm{O}(c, p) \cap \mathrm{O}(c, \mathrm{O}(p, q)) \cap \mathrm{O}(c, q)^{\mathrm{c}} \to \mathrm{O}(c, q)$ amounts to $\mathrm{E} \cap \mathrm{E} \cap c \to c^{\mathrm{c}}$, which is not true.)

That is, if our person also knows that q is not supported by that context, that observation is not withdrawn and, moreover, the situation is not considered to be inconsistent (by that person). However :

iv) $\dfrac{c \vdash p, \quad c \vdash p \to q, \quad c \vdash \neg q}{c \vdash q}$ is topologically valid, as well as

v) $\dfrac{c \vdash p, \quad c \vdash p \to q, \quad c \vdash \neg q}{c \vdash \bot}$, although

vi) $\dfrac{c \vdash p, \quad c \vdash p \to q, \quad c \vdash \neg q, \quad c \nvdash q}{c \vdash q}$ is not topologically valid.

Finally, for the sake of completeness,

vii) $\dfrac{c \vdash p, \quad c \vdash \neg p \vee q, \quad c \nvdash q}{\bot}$ is topologically valid.

(Proofs: iv) and v)): use the open-lemma and ii)).
vi): In the Euclidean plane E, let c denote a line, $p = \mathrm{E}$ and $q = c^{\mathrm{c}}$. Then

$$O(c, p) \cap O(c, O(p, q)) \cap O(c, q^c) \cap O(c, q)^c \to O(c, q)$$

amounts to $E \cap E \cap E \cap c \to c^c$, which is not true.

vii): For every topological space X, and all $c, p, q \subseteq X$,

$$O(c, p) \cap O(c, p^c \cup q) \subseteq O(c, q).$$

Hence $O(c, p) \cap O(c, p^c \cup q) \cap O(c, q)^c \to \varnothing$.)

11.4 Example (The "meaning" of implication for our imaginary person)

i) $\dfrac{c \vdash q}{c \vdash p \to q}$ is not topologically valid.

(Proof: If X is the real line, c the set of rational numbers, $p = X$ and $q = c$, then $O(c, q) \to O(c, O(p, q))$ amounts to $X \to \varnothing$, which is not true.)

Apparently, our imaginary person does not appreciate the proof step

$$\frac{q}{p \to q}$$

Not even "as a rule". On the other hand,

ii) $\dfrac{c \vdash q}{c \vdash (c \wedge p \to q)}$ is topologically valid.

(Proof: By Theorem 9.13, $\dfrac{c \vdash q}{c \wedge p \vdash q}$ is valid. By 11.3 i), $\dfrac{c \wedge p \vdash q}{c \vdash (c \wedge p \to q)}$ is valid. The open-lemma provides the rest.)

The resulting notion of implication ("$\to$", as treated by our imaginary person) seems to be somewhat different from the conception of implication that is common in mathematics. In mathematical reasoning, an utterance "a implies b", when done in a certain context, does not actually mean "a implies b". It means: "a, together with everything known or assumed so far, implies b". Our person seems to think of "$\to$" as having a different meaning. Our person seems to insist on being explicit about hidden assumptions : knowing "this is a penguin", the statement "if this is a bird, then it is a penguin" would not necessarily be accepted. On the other hand "if this is a bird and a penguin, then it is a penguin" would be considered true. This is affirmed by the following:

iii) $\dfrac{c \vdash p \wedge q \to r, \quad c \vdash q}{c \vdash p \to r}$ is not topologically valid.

(Proof: If X is the real line, c the set of rational numbers, $p = X$ and $q = r = c$, then $O(c, O(p \cap q, r)) \cap O(c, q) \to O(c, O(p, r))$ amounts to $X \to \varnothing$, which is not true.)

Hence, the person does not accept a statement $p \to r$ as correct only on the grounds that both q and $p \wedge q \to r$ are true.

11.5 Theorem

$\dfrac{c \vdash p \to q}{c \vdash (p \wedge r \to q)}$ and $\dfrac{c \vdash p \to q, \quad c \vdash p \to r}{c \vdash p \to (q \wedge r)}$ are topologically valid.

In general, we have :

If a, b, c $\in$ L such that $\dfrac{a}{b}$ is topologically valid, then

$\dfrac{c \vdash a}{c \vdash b}$ is also topologically valid.

Proof: Suppose that $\dfrac{a}{b}$ is topologically valid.

Then, for every extended model (X, O, φ),

$\varphi(a) \to \varphi(b)$ is true in X, hence $O(\varphi(a), \varphi(b)) = X$, hence,

by Theorem 9.13, $O(\varphi(c) \cap \varphi(a), \varphi(b))$ is full in X, hence,

by the open-lemma :

$$O(\varphi(c), \varphi(a)) \to O(\varphi(c) \cap \varphi(a), \varphi(b)).$$

But $O(\varphi(c), \varphi(a)) \cap O(\varphi(c) \cap \varphi(a), \varphi(b)) \subseteq O(\varphi(c), \varphi(b))$.

Hence, $O(\varphi(c), \varphi(a)) \to O(\varphi(c), \varphi(b))$.

Hence, $(c \to a) \to (c \to b)$ is true in (X, O, φ).

□

Likewise : If $a_1, \dots, a_n$, b, c $\in$ L such that

$\dfrac{a_1, \dots, a_n}{b}$ is topologically valid, then

$\dfrac{c \vdash a_1, \dots, c \vdash a_n}{c \vdash b}$ is also topologically valid.

For example, the fact that

$$\frac{c \vdash p, \quad c \vdash p \to q}{c \vdash q}$$

is topologically valid, can be seen by (a proof which essentially follows) the following scheme:

$$\cfrac{\cfrac{c \vdash p, \quad c \vdash p \to q}{c \vdash p \wedge (p \to q)} \qquad \cfrac{\overline{p \wedge (p \to q) \vdash q}}{c \wedge (p \wedge (p \to q)) \vdash q}}{c \vdash q}$$

The converse of Theorem 11.5 is not true, not even if c is a basic formula not occurring in the formulas $a_1, \ldots, a_n$ or b :

11.6 Example

i) $\dfrac{p, \quad p \to q, \quad q \to r}{r}$ is not topologically valid.

(Proof : if $p = \ell$, $q = \mathrm{E}$, $r = \ell^c$, then $p \cap \mathrm{O}(p, q) \cap \mathrm{O}(q, r) \to r$ amounts to $\ell \cap \mathrm{E} \cap \mathrm{E} \to \ell^c$, which is not true.)

ii) $\dfrac{c \vdash p, \quad c \vdash p \to q, \quad c \vdash q \to r}{c \vdash r}$ is topologically valid.

Proof-scheme, applying the open-lemma and Theorem 11.5 :

$$\cfrac{c \vdash p \qquad \cfrac{c \vdash p \to q \quad c \vdash q \to r}{c \vdash p \to r}}{c \vdash r}$$

□

So much for our imaginary person. It will be clear, though, that the imaginary person will continue to be a major source of inspiration in the sequel. In the mean time, we cannot ignore the following alternative point of view. Let us think of an agent whose set of sentences is closed under $\wedge$, $\vee$, $\neg$ (having their usual meaning) and $\to$ (signifying implication-up-to-possible-exceptions). The meaning of $\to$ is thought to be such that it forces the agent's inference relation to be non-monotonic, in that, for every pair a, b of sentences in the agent's language, the assumptions $a \to b$, a lead to b as a conclusion, but together with $\neg b$, they do no longer lead to the conclusion that b is true. This phenomenon is then expressed by the fact that the rule

$$\frac{a \to b, \quad a}{b} \text{ is topologically valid, but } \frac{a \to b, \quad a, \quad \neg b}{b} \text{ is not.}$$

11.7 Example (Birds & penguins)
As a consequence of Theorem 10.3,

i) $\dfrac{b \to f, \ \ p \to b \wedge \neg f}{p \to \neg f}$ is topologically valid, but so is

ii) $\dfrac{b \to f, \quad p \to b \wedge \neg f}{p \to f}$.

As a matter of fact,

iii) $\dfrac{b \to f, \quad p \to b \wedge \neg f}{\neg p}$ is topologically valid (since $\mathrm{O}(a, \varnothing) \subseteq a^{c}$).

On the other hand,

iv) $\dfrac{b \to f, \quad p \to b \wedge \neg f, \quad p}{f}$ is not topologically valid, while

v) $\dfrac{b \to f, \quad p \to b \wedge \neg f, \quad p}{\neg f}$ is.

(Proof of iv) : if $b = \mathrm{E}$, $p = \ell$, $f = \ell^{c}$, then $\mathrm{O}(b, f) \cap \mathrm{O}(p, b \cap f^{c}) \cap p \to f$ amounts to $\mathrm{E} \cap \mathrm{E} \cap \ell \to \ell^{c}$, which is not true.
Proof of v) : For all X, and all $b, f, p \subseteq \mathrm{X}$, $\mathrm{O}(p, b \cap f^{c}) \cap p \to f^{c}$ is true. By the open-lemma, $\mathrm{O}(b, f) \cap \mathrm{O}(p, b \cap f^{c}) \cap p \to f^{c}$ is also true.)

These statements (that is, iii), iv) and v)) could be read as follows:

If an object is such that:

if it is a bird, then it can fly,

if it is a penguin, then it is a bird and it cannot fly,

then : the object is not a penguin.

But if (it satisfies these conditions and) it *is* a penguin,
then it cannot fly.

vi) $\dfrac{b \to f, \quad p \to b \wedge \neg f, \quad b}{f}$ is also topologically valid, but

vii) $\dfrac{b \to f, \quad p \to b \wedge \neg f, \quad b}{\neg f}$ is not.

Proof: vi) is a direct consequence of the open-lemma and the topological validity of

$$\frac{b \to f, \quad b}{f}.$$

Proof of vii): if b = E, $p = \ell$, $f = \ell^c$, then $O(b, f) \cap O(p, b \cap f^c) \cap b \to f^c$ amounts to $E \cap E \cap E \to \ell$, which is not true.

11.8 Example (Concerning the deduction principles)
Comparing 11.7 ii) to 11.7 iv), we notice that

$$\frac{c_1, \ldots, c_n}{a \to b} \quad \text{does not imply} \quad \frac{c_1, \ldots, c_n, a}{b}.$$

The reverse also fails:

i) $\dfrac{a, \quad b}{b}$ is topologically valid, but $\dfrac{b}{a \to b}$ is not.

(If $a = \ell^c$, $b = \ell$, then $b \to O(a, b)$ amounts to $\ell \to \varnothing$, which is not true.)

The two principles involved are called deduction principles, and their failure seems, at first sight, undesired. But, in view of the example in §1 (about John not being home at six o'clock), the deduction principles *should* not be true : Let p denote "it is six o'clock, now", and q: "John is home, now". Then

$\dfrac{p \to q, \quad p, \quad \neg q}{\neg q}$ is acceptable, but $\dfrac{p \to q, \quad \neg q}{p \to \neg q}$ is not, since

this latter inference would allow us to conclude "John is never home at six o'clock" from "John is always home at six o'clock and John is not home, now" (whether it is six o'clock, now, or not).
On the other hand,

$\dfrac{p \to q, \quad \neg q}{p \to q}$ is acceptable, but $\dfrac{p \to q, \quad p, \quad \neg q}{q}$ is not

(as explained in §1), because of the meaning of $p \to q$.

Despite failure of the deduction principles as formulated above, we do have:

ii) $\dfrac{a \wedge b \to c}{a \to \neg b \vee c}$ and $\dfrac{a \to \neg b \vee c}{a \wedge b \to c}$ are topologically valid (by Theorem 10.3).

iii) $\dfrac{a \to b}{\top \to (a \to b)}$ and $\dfrac{\top \to (a \to b)}{a \to b}$ are topologically valid.

(Proof: $\mathrm{O}(a, b) \cap \mathrm{X} \to \mathrm{O}(a, b)$, hence $\mathrm{O}(a, b) \subseteq \mathrm{O}(\mathrm{X}, \mathrm{O}(a, b))$.
And, of course, $\mathrm{O}(\mathrm{X}, \mathrm{O}(a, b)) \cap \mathrm{X} \to \mathrm{O}(a, b)$.)

On the other hand,

iv) $\dfrac{a \wedge b \to c}{a \to (b \to c)}$ is not topologically valid.

(Proof: Let X be the real number line with its usual topology, let a be the set of rational numbers, $b = \mathrm{X}$, $c = a$, then $\mathrm{O}(a \cap b, c) \to \mathrm{O}(a, \mathrm{O}(b, c))$ amounts to $\mathrm{O}(a, a) \to \mathrm{O}(a, \varnothing)$, i.e. to $\mathrm{X} \to \varnothing$, which is not true.)

Oddly enough,

v) $\dfrac{a \wedge b \to c}{a \to (a \wedge b \to c)}$ is topologically valid (see 11.3 i)).

Equally surprisingly (in view of iv) above),

vi) $\dfrac{a \to (b \to c)}{a \wedge b \to c}$ is topologically valid.

Proof-scheme : $$\dfrac{\dfrac{a \to (b \to c)}{a \wedge b \to (b \to c)} \quad \dfrac{b \wedge (b \to c) \to c}{a \wedge b \wedge (b \to c) \to c}}{a \wedge b \to c}$$

(For all $b, c \subseteq \mathrm{X}$, $\mathrm{O}(b \cap \mathrm{O}(b, c), c) = \mathrm{X}$. By Theorem 9.13, $\mathrm{O}(a \cap b \cap \mathrm{O}(b, c), c)$ is full in X, etc.)

11.9 Example (Concerning the rationality laws)
Regarding the rationality laws, as presented in §8:

i) $\dfrac{a \to b, \quad a \wedge b \wedge c \not\to \bot}{a \wedge c \to b}$ (R1) is not topologically valid.

Proof: In $\mathbb{C}$, the set of complex numbers, define $a = \mathbb{C}$, $b = \mathbb{C} \setminus \mathbb{Q}$, $c = \mathbb{R}$. Then $O(a, b) = \mathbb{C}$, $O(a \cap b \cap c, \varnothing) = \mathbb{R}^c$, $O(a \cap c, b) = \mathbb{R}^c$, hence $O(a, b) \cap O(a \cap b \cap c, \varnothing)^c \to O(a \cap c, b)$ amounts to $\mathbb{C} \cap \mathbb{R} \to \mathbb{R}^c$, which is not true.

□

i') $\dfrac{a \to b, \quad a \wedge c \not\to b}{a \wedge b \wedge c \to \bot}$ is not topologically valid.

Proof: With the same definitions as in i),
$O(a, b) \cap O(a \cap c, b)^c \to O(a \cap b \cap c, \varnothing)$ amounts to $\mathbb{C} \cap \mathbb{R} \to \mathbb{R}^c$, which is not true.

□

ii) $\dfrac{a \to b, \quad a \wedge c \not\to \neg b}{a \wedge c \to b}$ (R2) is not topologically valid.

Proof: With the same definitions as in i), $O(a \cap c, b^c) = \mathbb{R}^c$, $O(a \cap c, b) = \mathbb{R}^c$, hence $O(a, b) \cap O(a \cap c, b^c)^c \to O(a \cap c, b)$ amounts to $\mathbb{C} \cap \mathbb{R} \to \mathbb{R}^c$, which is not true.

□

ii') $\dfrac{a \to b, \quad a \wedge c \not\to b}{a \wedge c \to \neg b}$ is not topologically valid.

Proof: With the same definitions as in i),
$O(a, b) \cap O(a \cap c, b)^c \to O(a \cap c, b^c)$ amounts to $\mathbb{C} \cap \mathbb{R} \to \mathbb{R}^c$, which is not true.

□

iii) $\dfrac{a \to b, \quad a \not\to \neg c}{a \wedge c \to b}$ (R3) is topologically valid.

Proof: Suppose it is not valid. Then, for some topological space X, and for some $a, b, c \subseteq X$, $O(a, b) \cap O(a, c^c)^c \not\to O(a \cap c, b)$, hence:

$O(a \cap c, b)^c$ is somewhere dense in $O(a, b) \cap O(a, c^c)^c$, say:

$O(a \cap c, b)^c$ is dense in $O \cap O(a, b) \cap O(a, c^c)^c$, the latter being nonempty.

Then $O \cap O(a, b) \cap O(a, c^c)^c \subseteq O(a \cap c, b)^c$, since this latter set is both dense and closed in the former.

Now define $U := O \cap O(a, b)$, $V := O(a, c^c)^c$.

(So that $U \cap V \subseteq O(a \cap c, b)^c$, and $U \cap a \to b$.)

Then: Since $U \cap V \neq \varnothing$ (i.e., $U \not\subset O(a, c^c)$), we know that $U \cap a \not\to c^c$.

Hence, c is somewhere dense in $U \cap a$, say

$O' \subseteq U$, c dense in $O' \cap a$, and $O' \cap a \neq \emptyset$.

Then $O' \cap a \not\to c^c$, hence $O' \not\subset O(a, c^c)$, hence $O' \cap V \neq \emptyset$.

But $O' \cap V \subseteq U \cap V \subseteq O(a \cap c, b)^c$, hence

$O' \cap O(a \cap c, b)^c \neq \emptyset$.

But, since c is dense in $O' \cap a$ and $O' \cap a \to b$, we know that $O' \cap a \cap c \to b$, hence $O' \subseteq O(a \cap c, b)$, contradiction.

□

iii') $\dfrac{a \to b, \quad a \wedge c \not\to b}{a \to \neg c}$ is topologically valid.

Proof: Suppose it is not valid. Then, for some topological space X, and for some $a, b, c \subseteq X$, $O(a, b) \cap O(a \cap c, b)^c \not\to O(a, c^c)$, hence:

$O(a, c^c)^c$ is somewhere dense in $O(a, b) \cap O(a \cap c, b)^c$, say:

$O(a, c^c)^c$ is dense in $O \cap O(a, b) \cap O(a \cap c, b)^c$, the latter being nonempty.

Then $O \cap O(a, b) \cap O(a \cap c, b)^c \subseteq O(a, c^c)^c$, since this latter set is both dense and closed in the former.

Now define $U := O \cap O(a, b)$, $V := O(a \cap c, b)^c$.

(So that $U \cap V \subseteq O(a, c^c)^c$, and $U \cap a \to b$.)

Then: Since $U \cap V \neq \emptyset$, (i.e., $U \not\subset O(a \cap c, b)$), we know that

$U \cap a \cap c \not\to b$.

Hence, b^c is somewhere dense in $U \cap a \cap c$, say

$O' \subseteq U$, b^c dense in $O' \cap a \cap c$, and $O' \cap a \cap c \neq \emptyset$.

Then $O' \cap a \cap c \not\to b$, hence $O' \not\subset O(a \cap c, b)$, hence $O' \cap V \neq \emptyset$.

But $O' \cap V \subseteq U \cap V \subseteq O(a, c^c)^c$, hence

$O' \cap O(a, c^c)^c \neq \emptyset$.

Hence, $O' \cap a \not\to c^c$, say

$O'' \subseteq O'$, c is dense in $O'' \cap a$, and $O'' \cap a \neq \emptyset$.

$O'' \cap a \to b$, hence $O'' \cap a \cap c \to b$.

But b^c is dense in $O'' \cap a \cap c$, contradiction.

□

iv) $\dfrac{a \vee b \to c, \quad a \not\to c}{b \to c}$ (R4) is topologically valid.

Proof-scheme, using iii') :

$$\dfrac{\dfrac{a\vee b\to c \qquad \dfrac{\dfrac{a\vee b\to c \quad \dfrac{a\not\to c}{(a\vee b)\wedge a\not\to c}}{a\vee b\to\neg a}}{a\vee b\to b}}{(a\vee b)\wedge b\to c}}{b\to c}$$

□

v) $\dfrac{a\to b,\quad a\wedge c\not\to b}{a\wedge\neg c\to b}$ (R5) is topologically valid.

Proof-scheme, using, again, iii') :

$$\dfrac{\dfrac{a\to b \quad a\wedge c\not\to b}{a\to\neg c}}{a\wedge\neg c\to b}$$

□

vi) $\dfrac{a\to b,\quad a\wedge c\to\neg b}{a\to\neg c}$ (R6) is topologically valid as a consequence of Theorem 10.3.

Recapitulating : R3, R4, R5 and R6 are topologically valid, R1 and R2 are not. It is interesting to note that assuming the "ordinary" validity of R1 or R2 is generally thought to be unreasonably strong, while "ordinary" R3, R4 and R5 are sometimes defended as acceptable (for example, in [KLM 90] or [LM 92]).

11.10 Example (Concerning modus tollens)

i) $\dfrac{a\to b}{\neg b\to\neg a}$ is topologically valid, by Theorem 10.3.

ii) $\dfrac{a\to b,\quad \neg b}{\neg a}$ is not topologically valid: $O(E, \ell^c)\cap\ell\to\varnothing$ is not true (where E denotes the Euclidean plane, and ℓ a line in E).

iii) For all $a, b \in L$, if $\dfrac{a}{b}$ is topologically valid, then $\dfrac{b\to\bot}{a\to\bot}$ is also topologically valid.

Proof: We will prove that $a\to b$ implies $O(b,\varnothing)\to O(a,\varnothing)$, by proving that it implies $O(b,\varnothing)\subseteq O(a,\varnothing)$:

We know that $O(b,\varnothing)\cap b=\varnothing$. Hence, $O(b,\varnothing)\cap b\cap a=\varnothing$.

But b is dense in a, and $O(b,\varnothing)$ is an open set. Hence $O(b,\varnothing)\cap a=\varnothing$.

Hence, $O(b, \varnothing) \subseteq O(a, \varnothing)$.

$\square$

iv) $\dfrac{c \vdash p \to q \quad c \vdash q \to \bot}{c \vdash p \to \bot}$ is topologically valid, by Theorem 11.5.

v) $\dfrac{c \vdash p \to q \quad c \vdash \neg q}{c \vdash \neg p}$ is topologically valid. The core of the proof is represented by the following scheme :

$$\dfrac{\dfrac{\dfrac{\dfrac{c \vdash p \to q}{c \wedge p \vdash p \to q} \quad c \wedge p \vdash p}{c \wedge p \vdash q} \quad \dfrac{c \vdash \neg q}{c \wedge p \vdash \neg q}}{c \wedge p \vdash \neg p} \qquad c \wedge \neg p \vdash \neg p}{c \vdash \neg p}$$

vi) $\dfrac{c \vdash p \to q, \quad c \nvdash q}{c \nvdash p}$ is not topologically valid.

Proof: Let ℓ be a line in the Euclidean plane E. If $c = \ell$, $p = E$, $q = \ell^c$, then $O(c, O(p, q)) \cap O(c, q)^c \to O(c, p)^c$ amounts to $E \cap \ell \to \varnothing$, which is not true.

11.11 Example

i) $\dfrac{a \to b}{a \wedge c \to b}$ and $\dfrac{a \to b}{a \wedge \neg b \to b}$ are topologically valid (Theorem 10.3).

ii) $\dfrac{a \to b, \quad a, \quad c}{b}$ and $\dfrac{a \to b, \quad a, \quad \neg b}{b}$ are not.

(Proof of the latter : if $a = E$, $b = \ell^c$, then $O(a, b) \cap a \cap b^c \to b$ amounts to $E \cap E \cap \ell \to \ell^c$, which is not true. The former is not valid, by the substitution theorem.)

The rules i) and ii) above are somewhat disturbing. We will return to this issue in Chapter 5. Suffice it to say, at this moment, that for all $a, b \subseteq X$, $a \cap b^c \to b$ is equivalent to $a \subseteq b$. Hence, a person accepting the rule

$$\frac{a \to b}{a \wedge \neg b \to b}$$

as valid, could be said to accept, as a rule, the principle "if $a \to b$ is true, then typically it is true-without-exceptions". That is, if there is no reason to assume that an implicational statement has exceptions, it is assumed to have none. Although this sounds reasonable, the question may arise whether it is possible to find mathematical caricatures displaying different behaviour in this respect, while still featuring rules of inference with possible exceptions.

We end this section with the following useful theorem, generalizing many of the above examples.

11.12 Theorem Let $a_0, \dots, a_n, b_0, \dots, b_n$ be classical formulae.
Then the following two statements are equivalent :

i) the rule $\dfrac{a_1 \to b_1\,,\ \dots\ ,\ a_n \to b_n,\ a_0}{b_0}$ is topologically valid,

ii) the rule $\dfrac{a_1 \to b_1\,,\ \dots\ ,\ a_n \to b_n}{a_0 \to b_0}$ is a derived rule of P1-P5.

Proof:

i) $\Rightarrow$ ii) : Suppose that the rule $\dfrac{a_1 \to b_1\,,\ \dots\ ,\ a_n \to b_n}{a_0 \to b_0}$ is not a derived rule of P1-P5. Then, by the Completeness Theorem of §7 (Corollaries 7.6 and 7.10), there is a topological model (X, φ) such that

$a_1 \to b_1, \dots, a_n \to b_n$ are true, and $a_0 \to b_0$ is not.

Let (X, O, φ') be the (unique) extended model such that, for every basic formula α of L, $\varphi'(\alpha) = \varphi(\alpha)$.

Then $a_1 \to b_1, \dots, a_n \to b_n$ are true in (X, O, φ'), but

$a_0 \to b_0$ is not.

Hence, $\varphi'(a_i \to b_i) = X$, for $i = 1, \dots, n$, hence

$$\varphi'(a_1 \to b_1) \cap \dots \cap \varphi'(a_n \to b_n) \cap \varphi'(a_0) \to \varphi'(b_0)$$

amounts to $X \cap \dots \cap X \cap \varphi'(a_0) \to \varphi'(b_0)$, which is not true.

Hence, the rule $\dfrac{a_1 \to b_1\,,\ \dots\ ,\ a_n \to b_n,\ a_0}{b_0}$ is not topologically valid.

To understand ii) $\Rightarrow$ i) : Suppose that the rule $\dfrac{a_1 \to b_1\ ,\ \ \dots\ ,\ \ a_n \to b_n}{a_0 \to b_0}$ is a derived rule of P1-P5, and that (X, O, φ) is an extended model.
Define $O := \varphi(a_1 \to b_1) \cap \dots \cap \varphi(a_n \to b_n)$.
Then $O \cap \varphi(a_i) \ \to\ \varphi(b_i)$, for $i = 1, \dots, n$.
Since the rule above is a derived rule of P1-P5, it follows that

$$O \cap \varphi(a_0) \ \to\ \varphi(b_0),$$

which was to be proven.
To illustrate the crucial step in this argument by an example :

the rule $\dfrac{a \to b,\quad a \to c,\quad a,\quad b}{c}$ is topologically valid, since, defining $O := O(a, b) \cap O(a, c)$, $O \cap a \to b$ and $O \cap a \to c$, hence $O \cap a \cap b \to c$.

□

§12 Other Implication Operators

The properties of inference-as-a-rule-with-possible-exceptions, as found in the preceding three sections, rely on the choice of the implication operator. The operator O(,) that we used turned out to be quite suitable. In this section, we will consider some alternatives, and investigate their behaviour.

A particularly interesting construction is the following.

U-validity
Let X be a topological space.
12.1 Definition For all $a, b \subseteq X$,

$$U(a, b) = \{\ w \in X \mid \text{every } O \text{ such that } w \in O,\ O \cap a \neq \varnothing \text{ contains an } O' \text{ such that } w \in O',\ O' \cap a \neq \varnothing,\ O' \cap a \to b\ \}.$$

12.2 Proposition For all (X and all) $a, b \subseteq X$,
i) $U(a, b) = X$ (only) if $a \to b$, (U(,) is an implication operator)
ii) $U(a, b) = O(a, b) \cap \bar{a}$ if $a \not\to b$.
(Where $\bar{a}$ denotes the closure of a, see appendix B, in particular B.7.)
Proof:
i) If $a \to b$, then for every O : $O \cap a \to b$, hence $U(a, b) = X$.
On the other hand, if $U(a, b) = X$, then every O such that $O \cap a \neq \varnothing$ contains an O' such that $O' \cap a \neq \varnothing$ and $O' \cap a \to b$.

Then O' contains an O'' such that $O'' \cap a \neq \emptyset$ and $O'' \cap a \subseteq b$.
Hence $a \to b$.
ii)
If $w \in \bar{a}$ (that is, every O containing w contains elements of a), then

$w \in U(a, b)$ (only) if $w \in O(a, b)$:

If $w \in U(a, b)$, then $w \in X$, X is open and $X \cap a \neq \emptyset$,

hence, there is an $O \subseteq X$ such that

$w \in O, O \cap a \neq \emptyset$ and $O \cap a \to b$.

Hence, $w \in O \subseteq O(a, b)$.

On the other hand, if $w \in O(a, b)$, then every O such that

$w \in O$ and $O \cap a \neq \emptyset$ also satisfies

$w \in O \cap O(a, b), O \cap O(a, b) \cap a \neq \emptyset$ (since $w \in \bar{a}$),

and $O \cap O(a, b) \cap a \to b$.

Hence, $w \in U(a, b)$.

If $w \notin \bar{a}$, then $w \in U(a, b)$ (only) if $a \to b$,
which is seen as follows.

That $a \to b$ implies $w \in U(a, b)$ follows from i).

Now, suppose that $w \in U(a, b)$ and that $w \notin \bar{a}$.

Say, $w \in O$ and $O \cap a = \emptyset$.

Let O' be such that $O' \cap a \neq \emptyset$.

Then $w \in O \cup O'$ and $(O \cup O') \cap a \neq \emptyset$.

Since $w \in U(a, b)$, there is an $O'' \subseteq O \cup O'$ such that

$w \in O'', O'' \cap a \neq \emptyset$ and $O'' \cap a \to b$.

But $O \cap a = \emptyset$, hence $(O'' \cap O') \cap a \neq \emptyset$.

Moreover, $(O'' \cap O') \cap a \to b$. (And $O'' \cap O' \subseteq O'$.)

Hence, $a \to b$.

□

A closer look at the proof above yields the following alternative formulation of Proposition 12.2 :

12.3 Corollary For all $a, b \subseteq X$, and all $w \in X$,

$w \in U(a, b)$ (only) if

either $w \notin \bar{a}$ and $a \to b$,
or $w \in \bar{a}$ and $w \in O(a, b)$.

Hence, Definition 12.1 amounts to the following: if there are no a-worlds in "the" direct environment of w, then $a \to b$ is true in w (only) if it is true in the totality of all the worlds. If there are a-worlds arbitrarily close to w, then

$a \to b$ is true in w (only) if, within some environment of w, almost all a-worlds are b-worlds.

12.4 Definition A rule $\dfrac{a_1, \dots, a_m}{b_1, \dots, b_n}$ is called *U-valid* whenever the formula $a_1 \wedge \dots \wedge a_m \to b_1 \wedge \dots \wedge b_n$ is true in every extended model based on U(,) (that is, in every extended model (X, U(,), φ)).

12.5 Proposition $\dfrac{a \to b}{a \wedge c \to b}$ is not U-valid.

Proof: Let E be the Euclidean plane, ℓ a line, P a point on ℓ.
If $a = \ell$, $b = \ell \setminus \{P\}$ and $c = \{P\}$, then $a \to b$ is true, $a \cap c \to b$ is not.
Hence, $U(a, b) = X$, but $U(a \cap c, b) = O(a \cap c, b) \cap (\overline{a \cap c}) = \varnothing$, since $O(a \cap c, b) = O(a \cap c, \varnothing) = (\overline{a \cap c})^c$.

□

12.6 Definition A standard rule $\dfrac{a_1 \to b_1\,, \quad \dots \quad, \quad a_n \to b_n}{a_0 \to b_0}$ is called *strongly U-valid* whenever, for all topological models (X, φ) :

$$U(\varphi(a_1), \varphi(b_1)) \cap \dots \cap U(\varphi(a_n), \varphi(b_n)) \subseteq U(\varphi(a_0), \varphi(b_0)).$$

12.7 Proposition For every standard rule, the following statements are mutually equivalent:

i) the rule is a derived rule of P1-P5,
ii) the rule is strongly U-valid,
iii) the rule is U-valid.

(Sketch of the) proof : If $A_0, \dots, A_{n+1}$ are standard formulas, and

$\dfrac{A_1\,, \quad \dots \quad, \quad A_n}{A_0}$ is strongly U-valid, then $\dfrac{A_1\,, \quad \dots \quad, \quad A_n\,, \quad A_{n+1}}{A_0}$ is also strongly U-valid.

Hence, to see i) $\Rightarrow$ ii), it suffices to prove the strong U-validity of each of the rules below.

(P1) $\dfrac{}{a \to a}$ (P2) $\dfrac{a \to b}{a \to b \vee c}$ (P3) $\dfrac{a \to b, \quad a \wedge b \to c}{a \to c}$

(P4) $\dfrac{a \to b, \quad a \to c}{a \wedge b \to c}$ (P5) $\dfrac{a \to c, \quad b \to c}{a \vee b \to c}$

P1 and P2 are, obviously, strongly U-valid.

For P3 : it is easy to prove that for all X and all $a, b \subseteq X$,

$$O(a, b) \cap O(a \cap b, c) \subseteq O(a, c).$$

By Proposition 12.2 and the fact that

$$\overline{a \cap b} \subseteq \overline{a}, \text{ (see B.8)}$$

it follows that P3 is strongly U-valid.

That P4 and P5 are strongly U-valid, can be proved in a similar way.

ii) $\Rightarrow$ iii) is a trivial consequence of the definitions.

To see iii) $\Rightarrow$ i), suppose that the rule $\dfrac{a_1 \to b_1 \ , \ \ldots \ , \ a_n \to b_n}{a_0 \to b_0}$ is not a derived rule of P1-P5. Then, by the Completeness Theorem of §7 (Corollaries 7.6 and 7.10), there is a topological model (X, φ) in which

$a_1 \to b_1, \ldots, a_n \to b_n$ are true, and $a_0 \to b_0$ is not.

Define $X' := X \times [0, 1]$, equipped with product topology (see Appendix B).

Let (X', U(,), φ') be an extended model such that, for $i = 0, \ldots, n$,

$$\varphi'(a_i) = \varphi(a_i) \times \{0\}, \ \ \varphi'(b_i) = \varphi(b_i) \times \{0\}.$$

(Note that $\varphi(a_i) \times \{0\}$ is essentially the same space as $\varphi(a_i)$.)

Then, using Proposition 12.2 i) and ii) respectively, it is easy to see that

$U_{X'}(\varphi'(a_i), \varphi'(b_i)) = X'$, for $i = 1, \ldots, n$, while, on the other hand,

$U_{X'}(\varphi'(a_0), \varphi'(b_0))$ is not full in X', since

$$U_{X'}(\varphi'(a_0), \varphi'(b_0)) \subseteq \overline{\varphi'(a_0)} \subseteq X \times \{0\}.$$

Hence, the rule is not U-valid.

□

As an easy consequence of Proposition 12.2, we have :

12.8 Proposition $\dfrac{a \to b, \ a}{b}$ is U-valid.

Proof: For all X and all $a, b \subseteq X$:

If $a \to b$, then $U(a, b) \cap a = a = O(a, b) \cap a$.

If $a \not\to b$, then $U(a, b) \cap a = O(a, b) \cap \bar{a} \cap a = O(a, b) \cap a$.

But $O(a, b) \cap a \to b$.

Hence, $U(a, b) \cap a \to b$.

□

To recapitulate, a standard rule is

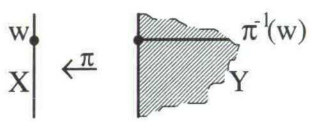
i) O-valid (only) if it is classically valid

ii) U-valid (only) if it is a derived rule of P1-P5.

For rules that are *not* standard, however, there is a considerable difference between topological validity (O-validity) and classical validity, as was seen in §11. It is hard to understand what it could mean for a non-standard rule to be "derived from P1-P5". One is inclined to say that no non-standard rule can be derived from P1-P5. On the other hand, U-validity of non-standard rules is a well-defined notion.

O_π-validity

Another possibility is to change the conception of a model, for example in the following way. Let X and Y be topological spaces, and π: Y $\rightarrow$ X a continuous map. We think of the elements of X as possible worlds and, for every w $\in$ X, of π^{-1}(w) as the collection of objects in that world. The topology on Y induces a topology on π^{-1}(w), for every w $\in$ X. Thus, the triple (π, X, Y) represents a "topological space of topological spaces".



Given such a triple, we could define, for all a, $b \subseteq$ Y,

$$T_\pi(a, b) := \pi^{-1}(\{ w \in X \mid \pi^{-1}(w) \cap a \rightarrow b \})$$

and evaluate sentences and rules in the extended model (Y, T_π, φ). Note that, if X is a one point space, then $T_\pi(\ , \ ) = T(\ , \ )$, the trivial implication operator, see Example 9.5.

In general, however, $T_\pi(\ , \ )$ is not an implication operator. (We will not prove this remark. It is an easy corollary of the proofs of 14.3, 14.4 and 14.5, see Chapter 4.) Hence, this construction is not entirely satisfying. A better operator to use, in combination with these triples, is the operator $O_\pi(,)$, defined as follows :

12.9 Definition For all a, $b \subseteq$ Y,

$O_\pi(a, b) := \pi^{-1}(O)$, where O is the largest open subset of X such that $\pi^{-1}(O) \cap a \rightarrow b$ is true (in Y).

A rule $\dfrac{a_1, \ldots, a_m}{b_1, \ldots, b_n}$ is called *O_π-valid* whenever, for all triples (π, X, Y) and all extended models (Y, O_π, φ), the formula

$$a_1 \wedge \ldots \wedge a_m \rightarrow b_1 \wedge \ldots \wedge b_n$$

is true in (Y, O_π, φ).

The rule is called *strictly O_π-valid* whenever, for all such extended models,

$$\varphi(a_1 \wedge \ldots \wedge a_m) \subseteq \varphi(b_1 \wedge \ldots \wedge b_n).$$

This construction generalizes both the trivial implication operator and O(,) : If $X = Y$ and $\pi(y) = y$ for all $y \in Y$, then $O_\pi(a, b) = O(a, b)$ for all $a, b \subseteq Y$. On the other hand, if X is a one point space (and π is the only existing map $Y \rightarrow X$), then $O_\pi(a, b) = T(a, b)$ for all $a, b \subseteq Y$.

Moreover $O_\pi(,)$ is, for every triple (π, X, Y), an implication operator on Y (which can be proved along the same lines as Proposition 9.8 ii).

12.10 Proposition

i) The rule $\dfrac{a \rightarrow b}{a \wedge c \rightarrow b}$ is not O_π-valid,

ii) A standard rule is O_π-valid (only) if it is a derived rule of P1-P5.

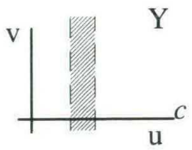
Proof: Although i) is a consequence of ii), we will give a direct proof:

Let Y be IR^2, $X = \mathrm{IR}$, $\pi(u, v) := u$, for all $(u, v) \in Y$.

Define $a = Y$, $b = \{ (u, v) \in Y \mid v \neq 0 \}$, $c = Y \setminus b$.

Then $a \rightarrow b$ is true in Y, hence $O_\pi(a, b) = Y$.

But, for any nonempty $O \subseteq \mathrm{IR}$, $(O \times \mathrm{IR}) \cap a \cap c \rightarrow b$ is not true.



Hence, $O_\pi(a \cap c, b) = \varnothing$. Hence,

$O_\pi(a, b) \rightarrow O_\pi(a \cap c, b)$ is not true, which proves i).

Proof of ii): As in Proposition 12.7, we will, for every standard rule, establish the equivalence of

1) the rule is O_π-valid,
2) the rule is strictly O_π-valid,
3) the rule is a derived rule of P1-P5.

3) $\Rightarrow$ 2) can be proved along the same lines as the corresponding part of Proposition 12.7. As an example, we will prove P3 to be strictly O_π-valid.

Let (π, X, Y) be a triple and $a, b, c \subseteq Y$. Let O_1 be the largest open subset of X such that $\pi^{-1}(O_1) \cap a \to b$. Let O_2 be the largest open subset of X such that $\pi^{-1}(O_2) \cap a \cap b \to c$. Then

$$\pi^{-1}(O_1 \cap O_2) \cap a = \pi^{-1}(O_1) \cap \pi^{-1}(O_2) \cap a \to c.$$

Hence, $O_1 \cap O_2$ is a subset of the largest open $O' \subseteq X$ such that

$$\pi^{-1}(O') \cap a \to c.$$

Hence, $\pi^{-1}(O_1 \cap O_2) \subseteq O_\pi(a, c)$. Hence

$$O_\pi(a, b) \cap O_\pi(a \cap b, c) = \pi^{-1}(O_1) \cap \pi^{-1}(O_2) =$$
$$= \pi^{-1}(O_1 \cap O_2) \subseteq O_\pi(a, c),$$

which proves that P3 is strictly O_π-valid.

2) $\Rightarrow$ 1) is trivial.

1) $\Rightarrow$ 3) : Suppose that $\dfrac{a_1 \to b_1\ , \quad \dots\ , \quad a_n \to b_n}{a_0 \to b_0}$ is a standard rule that is not a derived rule of P1-P5. Then, by the Completeness Theorem of §7 (Corollaries 7.6 and 7.10), there is a topological model (Y, φ) such that $\varphi(a_i) \to \varphi(b_i)$ is true in Y, for $i = 1, \dots, n$, while $\varphi(a_0) \to \varphi(b_0)$ is not true in Y (hence, $Y \neq \varnothing$).

Define $X := \{x\}$ for some x, and $\pi(y) = x$ for all $y \in Y$.

Using the triple (π, X, Y),

$$O_\pi(a_i, b_i) = T(a_i, b_i) = Y, \text{ for } i = 1, \dots, n,$$

while $O_\pi(a_0, b_0) = \varnothing$.

Hence, $\dfrac{a_1 \to b_1\ , \quad \dots\ , \quad a_n \to b_n}{a_0 \to b_0}$ is not O_π-valid.

□

It is possible to generalize this construction further, by using infinite rows

$$X_1 \xleftarrow{\ \pi_1\ } X_2 \xleftarrow{\ \pi_2\ } X_3 \xleftarrow{\ \pi_3\ } \dots$$

instead of triples $X \xleftarrow{\ \pi\ } Y$. We will not proceed in this direction, since this will not yield anything essentially new or interesting for the purposes of this thesis.

There is, however, one adaptation of O_π-validity that we cannot ignore. Replacing $O_\pi(\ ,\ )$ by $O(\ ,\ )$ in Definition 12.9, triples can also be used for an interesting "alternative definition" of O-validity :

12.11 Definition

A rule $\dfrac{a_1, \ldots, a_m}{b_1, \ldots, b_n}$ is called *O'-valid* whenever, for all triples (π, X, Y) and all extended models (Y, O, φ), the formula

$$a_1 \wedge \ldots \wedge a_m \rightarrow b_1 \wedge \ldots \wedge b_n$$

is true in (Y, O, φ).

12.12 Proposition A rule is O'-valid (only) if it is O-valid.
(Proof: Elementary.)

Definition 12.11 amounts to the following. An object $y \in Y$ in world $x = \pi(y) \in X$ satisfies $a \rightarrow b$ whenever, in some open set containing y, almost all objects that satisfy a do also satisfy b. It might be, of course, that every open set containing y contains objects from worlds different from y's world. That is, whether y satisfies $a \rightarrow b$ or not, might depend on the truth value of a and b for objects near y, in worlds near x. Note that this does not necessarily mean that y itself has equivalents in worlds different from x.

This construction seems closer to our mental picture of implication than the straight definition of O-validity, and might help our understanding of phenomena such as 11.7 iii), iv) and v). Moreover, it might improve one's appreciation of using "$\rightarrow$" to interpret inference as done in Definition 9.11. Thus, this construction reinforces the remarks made in §10.

S-validity

A possible disadvantage of the operator O(,) is the fact that it doesn't satisfy the deduction principles (i.e.: there is a difference between

$$\frac{a}{b \rightarrow c}, \qquad \frac{a, \ \ b}{c}, \qquad \frac{}{a \wedge b \rightarrow c}, \qquad \text{etc.).}$$

This forces us to be cautious when formulating "practical" reasoning problems as rules (see 11.7 and 11.8). There is no alternative operator that behaves better in this respect :

12.13 Proposition Let X be a topological space, and I(,) an operator on $\mathcal{P}$(X) such that

i) for all $a, b, c \subseteq X$, $a \rightarrow I(b, c)$ (only) if $a \cap b \rightarrow c$,

ii) for all $a, b \subseteq X$, $I(a, b) \subseteq X$.
(I(,) is not assumed to be an implication operator.)
Then for all $a, b \subseteq X$, $I(a, b) = a^c \cup b$, and X is monotonic.
Proof: If I(,) satisfies i), then for all $p \in X$, and all $a, b \subseteq X$,

$$\{p\} \to I(a, b) \text{ (only) if } \{p\} \cap a \to b.$$

But $\{p\} \to I(a, b)$ (only) if $p \in I(a, b)$,
and $\{p\} \cap a \to b$ (only) if $p \in a^c \cup b$.

Hence, for all $a, b \subseteq X$, $I(a, b) = a^c \cup b$.
By i) it holds that (for all $a, b, c \subseteq X$)

$$a \to b^c \cup c \text{ (only) if } a \cap b \to c.$$

By Proposition 5.15, X is monotonic.
□

As a matter of fact, in connection with the idea behind the deduction principles, the operator O(,) behaves correctly, in the following sense.
If X is a set of possible worlds, and a, b, c are propositions, then restricting one's attention to the worlds satisfying a, the extension (w.r.t. this collection) of the sentence "$b \to c$" equals the extension of "$a \wedge b \to c$" w.r.t. the whole collection of possible worlds. In other words, a world satisfying a satisfies "$b \to c$" w.r.t. the a-worlds (only) if it satisfies "$a \wedge b \to c$" w.r.t. all possible worlds :

12.14 Proposition Let X be a topological space, and $a, b, c \subseteq X$.
Let $O_a(a \cap b, a \cap c)$ denote the largest $O \subseteq a$ (open in a) such that

$$O \cap a \cap b \to c.$$

Then $(a \cap)\ O_a(a \cap b, a \cap c) \ = \ a \cap O(a \cap b, c)$.
Proof : There is an O (open in X) such that

$$O \cap a = O_a(a \cap b, a \cap c),$$

hence $O \cap a \cap b \to c$, hence $O \subseteq O(a \cap b, c)$.
Hence, $O_a(a \cap b, a \cap c) \ \subseteq \ a \cap O(a \cap b, c)$.
On the other hand,

$$a \cap O(a \cap b, c) \cap a \cap b \to a \cap c, \text{ and}$$

$$a \cap O(a \cap b, c) \text{ is open in } a.$$

Hence, $a \cap O(a \cap b, c) \ \subseteq \ O_a(a \cap b, a \cap c)$.
□

Propositions 12.13 and 12.14 notwithstanding, one might still wonder whether there exist implication operators (on non-monotonic spaces) that validate the following two rules :

$$\frac{a \wedge b \to c}{a \to (b \to c)} \qquad\qquad \frac{a \to (b \to c)}{a \wedge b \to c}$$

(The latter is O-valid, but the former is not, see 11.8 iv) and vi).)

The answer to this question is yes, since, for every *finite* space, there is such an implication operator, as will be proved below, and there exist (many) non-monotonic finite topological spaces, by Theorem 7.5.

12.15 Definition Let X be a finite topological space. For all $a, b \subseteq X$, $S(a, b)$ denotes the union of all $S \subseteq X$ (not necessarily open) such that $S \cap a \to b$. A rule

$\dfrac{a_1, \dots, a_m}{b_1, \dots, b_n}$ is called *S-valid* whenever the formula $a_1 \wedge \dots \wedge a_m \to b_1 \wedge \dots \wedge b_n$ is true in every extended model (X, S, φ) with finite space X.

12.16 Proposition Let X be a finite topological space. Then :

i) For all $a, b \subseteq X$, $a^c \cup b \subseteq S(a, b)$,

ii) For all $a, b \subseteq X$, $S(a, b) \cap a \to b$
(i.e., $S(a, b)$ is *the* largest $S \subseteq X$ such that $S \cap a \to b$),

iii) $S(\ ,\ )$ is an implication operator on X.

Proof : i) is true, because $(a^c \cup b) \cap a \subseteq b$, hence $(a^c \cup b) \cap a \to b$.
To see ii), note that $S_1 \cap a \to b$, $S_2 \cap a \to b$ implies $(S_1 \cup S_2) \cap a \to b$, and that $S(a, b)$ is a union of finitely many sets.
iii) is an easy consequence of Definition 12.15 and ii) above.

□

12.17 Proposition The rule $\dfrac{a \to b}{a \wedge c \to b}$ is S-valid.

Proof : For all X, and all $a, b, c \subseteq X$, $S(a, b) \cap a \to b$.
Hence (5.12 iv)), $S(a, b) \to a^c \cup b$.
But $a^c \cup b \subseteq (a \cap c)^c \cup b \subseteq S(a \cap c, b)$, by 12.16 i).
Hence, $S(a, b) \to S(a \cap c, b)$.

□

12.18 Proposition The rules $\dfrac{a \wedge b \to c}{a \to (b \to c)}$ and $\dfrac{a \to (b \to c)}{a \wedge b \to c}$ are S-valid.

Proof : For all X, and all $a, b, c \subseteq X$,

i) $S(a \cap b, c) \cap a \cap b \to c$, hence $S(a \cap b, c) \cap a \to b^c \cup c$.

But $b^c \cup c \subseteq S(b, c)$.

Hence, $S(a \cap b, c) \cap a \to S(b, c)$.

Hence, $S(a \cap b, c) \subseteq S(a, S(b, c))$.

ii) $S(a, S(b, c)) \cap a \to S(b, c)$,

hence $S(a, S(b, c)) \leftrightarrow a^c \cup S(b, c)$.

But $a^c \subseteq (a \cap b)^c \cup c \subseteq S(a \cap b, c)$, by 12.16 i),

and $S(b, c) \to S(a \cap b, c)$, by Proposition 12.17.

Hence, $a^c \cup S(b, c) \to S(a \cap b, c)$.

By Proposition 5.12 ii),

$$S(a, S(b, c)) \to S(a \cap b, c).$$

□

At first sight, the operator S(,) looks more natural than O(,), but it has some annoying properties :

12.19 Proposition The rule $\dfrac{\mathrm{a} \to \mathrm{b}, \quad \mathrm{a} \to \mathrm{c}}{\mathrm{a} \to \mathrm{b} \wedge \mathrm{c}}$ is not S-valid.

Proof : Let X be $\{1, 2\}$, with a topology having as open sets: $\varnothing$, X and $\{1\}$.
Define $a = X$, $b = \{1\}$, $c = \{2\}$.
Then $S(a, b) = a$, $S(a, c) = c$ and $S(a, b \cap c) = \varnothing$.
Hence, $S(a, b) \cap S(a, c) \to S(a, b \cap c)$ is not true in X.

□

(Of course, $S(a, b) = X$ and $S(a, c) = X$ does imply $S(a, b \cap c) = X$.)

Hence, despite Proposition 12.17, S-validity could not have been used to obtain Theorem 9.15.

T-validity

In 9.5 the trivial implication operator, T(,), was defined. The main advantage of this operator is that T-validity of formulae is tightly connected with mathematical properties of the relation "$\to$", as we will see below.

12.20 Definition Let X be a topological space. For all $a, b \subseteq X$,

$T(a, b) := X$ if $a \to b$, and $T(a, b) := \varnothing$ if $a \not\to b$.

A *formula* of L is called *T-valid* whenever it is true in every extended model based on T(,). A rule

$$\frac{a_1, \dots, a_m}{b_1, \dots, b_n}$$

is *T-valid* whenever the formula $a_1 \wedge \dots \wedge a_m \rightarrow b_1 \wedge \dots \wedge b_n$ is T-valid.

Note that if (X, T, φ) is an extended model based on T(,), and $\varphi(a_i) = a_i$, for $i = 1, \dots, 4$, then, say, $\neg(a_1 \rightarrow a_2) \vee (a_3 \rightarrow a_4)$ is true in X (only) if $\varphi(a_1 \rightarrow a_2) = \varnothing$ or $\varphi(a_1 \rightarrow a_2) = X$. (For all $a, b \subseteq X$, $T(a, b)$ is either X or $\varnothing$. Hence, if $a \rightarrow b$ is a standard formula, then $\varphi(a \rightarrow b)$ is either X or $\varnothing$.) Hence, $\neg(a_1 \rightarrow a_2) \vee (a_3 \rightarrow a_4)$ is true (only) if $a_1 \not\rightarrow a_2$ or $a_3 \rightarrow a_4$. Likewise, it is easy to see that *a standard rule is T-valid (only) if it is a derived rule of P1-P5.*
In particular, we have:

12.21 Proposition The rule $\dfrac{a \rightarrow b}{a \wedge c \rightarrow b}$ is not T-valid.

Hence, T-validity does not allow a result like Theorem 9.15. The construction is of interest, nevertheless, because of an appealing connection with the O-operator (Theorem 12.24, below).

12.22 Definition A formula of L is called *O-valid* whenever it is true in every extended model based on O(,). A rule

$$\frac{a_1, \dots, a_m}{b_1, \dots, b_n}$$

is called *strictly O-valid* whenever, for all extended models based on O(,),

$$\varphi(a_1 \wedge \dots \wedge a_m) \subseteq \varphi(b_1 \wedge \dots \wedge b_n).$$

12.23 Definition The *order* of a formula of L is defined inductively as follows. For all $a, b \in L$:

order(a) = 0 if a is a basic formula,
order($\neg$a) = order(a),
order(a $\wedge$ b) = order(a $\vee$ b) = max {order(a), order(b)},
order(a $\rightarrow$ b) = max {order(a), order(b)} + 1,

Instead of "order(a) = n" we just say "a is a formula of order n".
Note that the formulae of order 0 are precisely the classical formulae of Definition 10.1.

The essence of the following theorem was stated in [Boutilier 89] in the context of preferential semantics. We give a straightforward topological reconstruction of the proof found in [Boutilier 94b].

12.24 Boutilier's Theorem Let $a \in L$ be of order 1. Then

a is O-valid (only) if a is T-valid.

Proof: We will prove, for every $a \in L$ of order 1,

$\neg a$ is not O-valid (only) if $\neg a$ is not T-valid,

by proving:

there is an extended model (X, O, φ) such that $\varphi(a) \neq \emptyset$ (only) if
there is an extended model (X', T, φ') such that $\varphi'(a) \neq \emptyset$.

Since a is of order 1, it is a Boolean combination of classical formulae and conditionals of classical formulae. But "$\alpha_1 \vee \ldots \vee \alpha_k$ is not O-valid" implies that at least one of $\alpha_1, \ldots, \alpha_k$ is not O-valid. Likewise, "$\alpha_1 \vee \ldots \vee \alpha_k$ is not T-valid" implies that one of $\alpha_1, \ldots, \alpha_k$ is not T-valid. Hence, without loss of generality, we may assume :

$$a \equiv (a_1 \to b_1) \wedge \ldots \wedge (a_n \to b_n) \wedge \neg(a_{n+1} \to b_{n+1}) \wedge \ldots \wedge \neg(a_{n+m} \to b_{n+m}) \wedge a_0$$

where $a_0, \ldots, a_{n+m}, b_1, \ldots, b_{n+m}$ are classical formulae.

1) Suppose that (X, O, φ) is an extended model such that $\varphi(a) \neq \emptyset$.
Say, $w \in \varphi(a)$.
Define $U := \varphi((a_1 \to b_1) \wedge \ldots \wedge (a_n \to b_n))$.
Then U is an open subset of X.
Let $(U, T(,), \varphi')$ be the (unique) extended model such that,
for every basic formula α of L :

$$\varphi'(\alpha) = \varphi(\alpha) \cap U.$$

Then, for $i = 1, \ldots, n$:

$$U \cap \varphi(a_i) \to \varphi(b_i),$$

hence $\quad a_i \to b_i$ is true in (U, T, φ'),

hence $\quad \varphi'(a_i \to b_i) = U$.

On the other hand, for $j = 1, \ldots, m$:

$$w \notin \varphi(a_{n+j} \to b_{n+j}) \text{ and } w \in U,$$

hence, $\quad U \cap \varphi(a_{n+j}) \not\to \varphi(b_{n+j})$,

hence, $\quad \varphi'(a_{n+j} \to b_{n+j}) = T(\varphi'(a_{n+j}), \varphi'(b_{n+j})) = \emptyset$.

Since $\quad w \in \varphi(a_0) \cap U = \varphi'(a_0)$,

$$\varphi'(a) \neq \emptyset.$$

Hence, (U, T, φ') is an extended model such that $\varphi'(a) \neq \emptyset$.

2) Suppose that (X, T, φ) is an extended model such that $\varphi(a) \neq \varnothing$.
Say, $w \in \varphi(a)$.
Choose any object w' such that $w' \notin X$, and define :

$$X' := X \cup \{w'\}.$$

A topology on X' is defined as follows:

$\nu \subseteq X'$ is open in X' whenever
either $\nu = X'$
or $\nu \subseteq X$ and ν is open in X.

It is easy to check that this defines a topology on X', and that the topology on X equals the topology (on X) induced from X'.
Let $(X', O(,), \varphi')$ be the (unique) extended model such that, for every basic formula α of L :

$$\varphi'(\alpha) \cap X = \varphi(\alpha)$$

and $w' \in \varphi'(\alpha)$ (only) if $w \in \varphi(\alpha)$.

(The above construction amounts to adding a "copy" of w to the space X : w' satisfies the same basic formulas as w.)

As before, we assume that :

$$a \equiv (a_1 \to b_1) \wedge \ldots \wedge (a_n \to b_n) \wedge \neg(a_{n+1} \to b_{n+1}) \wedge \ldots \wedge \neg(a_{n+m} \to b_{n+m}) \wedge a_0.$$

Note that $w' \in \varphi'(a_0)$, since $w \in \varphi(a_0)$.
Moreover, for all $a, b \subseteq X'$,

$$w' \in O(a, b) \text{ (only) if } X' \cap a \to b$$

(since X' is the only open subset containing w').
Hence (as is straightforward to check), for i = 1, ... , m+n,

$$w' \in O(\varphi'(a_i), \varphi'(b_i)) \text{ (only) if } a_i \to b_i \text{ is true in } (X, T, \varphi).$$

Hence,

$w' \in \varphi'(a_i \to b_i)$ for i = 1, ... , n,
$w' \notin \varphi'(a_{n+j} \to b_{n+j})$ for j = 1, ... , m :

and $w' \in \varphi'(a_0)$.
Hence, $w' \in \varphi'(a)$, and (X', O, φ') is an extended model such that $\varphi'(a) \neq \varnothing$.
□

There is a number of remarks to be made about this theorem.
First, note that the formula $((a \to b) \to (a \wedge c \to b))$ is O-valid, but not T-valid (Proposition 12.21). Hence, Boutilier's theorem cannot be extended to formulas of higher order.
Furthermore, although this theorem may seem to be, at first sight, a strengthening of Theorem 10.3, it constitutes an entirely different result. For

example, Boutilier's theorem cannot be used to prove the topological validity of the rule of monotony, as this amounts to proving the O-validity of the (second order) formula $((a \to b) \to (a \wedge c \to b))$.
Boutilier's theorem may also seem, at first sight, an abstract, technical peculiarity. The following example, however, will illustrate its applicability.

Let us first recapitulate some definitions.
A rule $\dfrac{a}{b}$ is O-valid (only) if (the formula) $a \to b$ is O-valid.

A rule $\dfrac{a}{b}$ is strictly O-valid (only) if $\neg a \vee b$ is O-valid.

In 11.9 iii), we saw that the rule

$$\frac{a \to b, \quad a \not\to \neg c}{a \wedge c \to b}$$

is O-valid. Boutilier's theorem helps us to prove that it is not strictly O-valid. The formula

$$\neg((a \to b) \wedge \neg(a \to \neg c)) \ \vee \ (a \wedge c \to b))$$

is a first order formula. Hence, it is O-valid (only) if it is T-valid, hence (only) if, for every topological space X, and all $a, b, c \subseteq X$,

$$T(a, b) = X,\ T(a, c^c) = \varnothing \text{ implies } T(a \cap c, b) = X, \text{ that is :}$$

$$a \to b,\ a \not\to c^c \text{ implies } a \cap c \to b.$$

This latter statement, however, is not true, as we saw in Example 6.6.

In this way, Boutilier's theorem can be used to prove a similar result about any rule

$$\frac{a_1, \quad \ldots, \quad a_n}{b_1, \quad \ldots, \quad b_m}$$

where $a_1, \ldots, a_n, b_1, \ldots, b_m$ are first order formulae.
In particular, we can prove in this way that a standard rule is T-valid (only) if it is strictly O-valid. And as a corollary of this :

12.25 Corollary A standard rule is strictly O-valid (only) if it is a derived rule of P1-P5.

Note that these remarks (in particular Corollary 12.25) reinforce the interpretation of O-validity of standard rules as validity-up-to-possible-exceptions (see §10).

Recapitulating §12

Neither U-validity, nor O_π-validity, nor S-validity, nor T-validity could have been used to obtain Theorem 9.15. O'-validity is equivalent to O-validity.

§13 Comparison with Other Approaches

At the end of Chapter 2, the reader was urged to forget, for the rest of this thesis, about any association of "$\rightarrow$" with preference of some possible worlds over others. We will now have to return to that issue, though only temporarily, in order to compare our constructions with some of the already existing constructions.

If $(X, \leq)$ is a partial ordering (of possible worlds), we may define,
for $a, b \subseteq X$,

$$\mathrm{Dc}(a, b) := \{\, w \in X \mid \underset{\wedge}{w} \cap a \mathrel{|\!\sim}_{\leq} b \,\}$$

and $$\mathrm{Uc}(a, b) := \{\, w \in X \mid \overset{\vee}{w} \cap a \mathrel{|\!\sim}_{\leq} b \,\}$$

where, for every $w \in X$,

$$\underset{\wedge}{w} := \{\, v \in X \mid v \leq w \,\} \quad \text{and} \quad \overset{\vee}{w} := \{\, v \in X \mid w \leq v \,\},$$

called the *downcone* and the *upcone* of w, respectively.
($\mathrel{|\!\sim}_{\leq}$ was defined in 7.3).

Each of these operators can be used to evaluate (sentences containing) nested conditionals in preferential models. In [Makinson 93], this is referred to as the *downcone construction* and the *upcone construction*, respectively. It will be clear that it is also possible to define downcone-validity and upcone-validity of *rules*, similar to our definition of O-validity, U-validity, etc.

The operator Uc(,) is, in general, not an implication operator. For, if
$(X, \leq)$ is the following partial ordering :

$a = \{w_1, w_3\}$ and $b = \{w_2, w_3\}$, then $a \vdash_{\leq} b$, but $w_2 \notin \mathrm{Uc}(a, b)$, hence $\mathrm{Uc}(a, b) \neq X$. Moreover, $X \nvdash_{\leq} \mathrm{Uc}(a, b)$.
Hence, the upcone construction seems to be not very appropriate.

The downcone construction, on the other hand, is much more interesting. The operator O(,), as defined in 9.6, generalizes the downcone construction, in the following sense : if $(X, \leq)$ is a partial ordering, we may define, in accordance with the proof of Theorem 7.5,

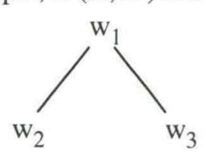
$$a \subseteq X \text{ is open whenever } \hat{w} \subseteq a \text{ for all } w \in a .$$

It is now easy to see that, with respect to this topology on X,

$$O(a, b) = \mathrm{Dc}(a, b) \text{ for all } a, b \subseteq X.$$

Hence, every downcone operator is the O(,) operator of some topology.
As a corollary, the downcone construction shares all phenomena found for O(,), in particular those indicated in 9.13, 9.15 and §10. Although the downcone-validity of modus ponens and the rule of monotony was mentioned in [Boutilier 89], Theorems 9.15, 10.3, 11.5, 11.12 and Example 10.9 were never noticed before, as far as I know.

Moreover, our results are more general, in that not every O(,) operator is the downcone operator of some partial ordering, not even on spaces for which there does exist a partial ordering providing the right consequence relation. For example, if $(X, \leq)$ is the following partial ordering :



Then the definition above yields the topology

$$\tau_1 = \{\emptyset, \{w_2\}, \{w_3\}, \{w_2, w_3\}, X \},$$

which has the property that (for all $a, b \subseteq X$) $a \to b$ (only) if $a \vdash_{\leq} b$.
But the topology $\tau_2 := \{\emptyset, \{w_2, w_3\}, X \}$ also has this property.
If we let $O_i(,)$ denote the O-operator that is based on the topology τ_i, for i = 1, 2, then $O_1(,)$ equals Dc(,), but $O_1(,)$ does not equal $O_2(,)$.
(For example, $O_1(\{w_2\}, \emptyset) = \{w_3\}$, but $O_2(\{w_2\}, \emptyset) = \emptyset$.)
Since $\leq$ is the only partial ordering on X yielding this consequence relation, there is no (other) partial ordering on X for which the downcone operator equals $O_2(,)$. Thus, $O_2(,)$ is not a downcone operator on the set X.

In general: on a single preferential model, there typically exist several different topologies providing the right consequence relation, each one leading to an O(,) operator, one of them being the downcone operator. Summarizing, we could say that, in §9, §10 and §11, we found some new results about the downcone construction, and a number of other operators displaying the same behaviour. It is important, though, to note the following as well.

Makinson criticizes the downcone construction on the grounds that "like the upcone one, (it) does not correspond closely to the basic idea underlying normality semantics". In other words, the downcone construction is not well motivated for preferential semantics. From our topological perspective, however, the operator O(,) is very natural; O(a, b) is the largest region in which $a \rightarrow b$ holds, where it is taken for granted that a region is an open set. In a preferential model, however, this identification is not so natural. Essentially, because the topological space associated with a preferential model will typically be a "pathological" space: a structure that happens to satisfy the requirements of a topological space without being a "carrier of geometrical intuition". In short, the downcone construction is better motivated by our topological presentation than by preferential semantics.

One more (important) reason to prefer the topological intuition over the preferential idea, is the following. To appreciate the interpretative remarks of §10, it is necessary to appreciate the non-standard conditional as a plausible interpretation of inference. While preferential semantics depicts the defeasible conditional as something that takes only the most normal possibilities into account (which may be a negligible minority among all possibilities), the topological semantics pretends to take every possibility into account, be it in a not too pedantic way. It is not claimed that our construction is entirely convincing as a plausible interpretation of inference. But the downcone construction (that is, the preferential interpretation of the operator O(,)), does not support the remarks of §10 at all.

In [Gabbay 95], a "fibring" construction was used to evaluate nested conditional sentences. This construction, unlike the downcone construction, does correspond to the idea underlying normality semantics. Gabbay's construction resembles T_π-validity, being more general in one respect and less general in another. Gabbay uses a "multi-layer" construction (which corresponds to our suggestion of using infinite rows of spaces and maps instead of triples). On the other hand, the preferential idea does not support any relationship between objects in one world and objects of another, while in a triple (π, X, Y), there is one topology on Y, instead of a separate topology on $\pi^{-1}(w)$ for every $w \in X$. For topological semantics, this latter generalization seems unproblematic. But $T_\pi(\, , \,)$, as said, is not an implication-operator, in general. Its adaptation, $O_\pi(\, , \,)$, does not validate the rule of monotony and is therefore unsuitable to obtain results like Theorem 9.15.

Some existing approaches towards nested defeasible conditionals are strongly connected with modal logic. For example, the main advantage of the downcone construction is its connection with the modal system S4. This connection has a topological equivalent. We will not give a detailed exposition, since it is a straightforward adaptation of the connection as described in [Boutilier 89]. It is based on the fact that for all topological spaces, X, and all $a, b \subseteq X$,

$$O(a, b) = \left((a^c)^o \cup \overline{(a \cap (a^c \cup b)^o)}\right)^o$$

(where a^o and $\bar{a}$ denote the interior and the closure of a, respectively).
As is well-known, the calculus of $\cap$, $\cup$, ^{c} and ^{o} in topological spaces is equivalent to the modal system S4, ^{o} playing the role of the $\Box$-modality.
Hence, the formula above gives rise to a translation from L into the modal language of S4 such that every formula of L is O-valid (see 12.22) (only) if its translation is a tautology of S4. (As a corollary of Boutilier's work and the remarks above, rules are downcone-valid (only) if they are topologically valid.)

This and similar connections have lead some people ([Lamarre 91], [Boutilier 94]) to tackle the task of finding suitable strengthenings of P1-P5 (see §8) by studying extensions of S4. For example, [Boutilier 89] pointed out that joining R3 to P1-P5 corresponds to using the modal system known as S4.3. To incorporate the most important adjustments, however, this approach will not be general enough.
For example, if $\dfrac{a \rightarrow b}{a \wedge c \rightarrow b}$ is to be valid, but $\dfrac{a \rightarrow b}{a \wedge \neg b \rightarrow b}$ is not,
or if $\dfrac{a \rightarrow b, \quad a, \quad \neg b}{b}$ is to be invalid, but $\dfrac{a \rightarrow b, \quad a, \quad c}{b}$ is to be valid
(c being a basic formula not occurring in a or b), we will have to consider modal systems that do not satisfy the substitution theorem. Since very little

is known about such modal systems, it seems useless to proceed in this direction. Likewise, it will be of little use to search for more convincing implication-operators, since this would necessarily lead to systems that do satisfy the substitution theorem.

Conclusions of Chapter 3

In this chapter it has been shown that there exist non-monotonic formalisms in which modus ponens and the rule of monotony are valid-up-to-possible-exceptions, as well as all other laws of classical propositional logic. As a consequence, the rules of classical propositional logic do not determine the meaning of deducibility and inference as implication-without-exceptions.

Chapter 4
Universal Quantification up to Possible Exceptions

The notion of a full subset will be used to interpret sentences involving a non-strict universal quantifier. The material presented is to be seen, in the first place, as preliminary for the next chapter, although the subject is of interest on its own.

§14 For practically all ...

The notion of a full subset arose, in §5, as an attempt to capture an intuitive, geometrical idea by a mathematical notion. Given a topological space X and $a, b \subseteq X$, we read $a \to b$ as "the elements of a are in b, up to possible exceptions". So far, we thought of such phrases as non-standard interpretations of implicational statements. However, we could just as well read $a \to b$ as "practically all elements of a are in b", and think of it as a non-standard interpretation of a statement involving a universal quantifier, namely "$\forall x \in a\ [\ x \in b\ ]$". In order to investigate this possibility, we will extend our repertoire of geometrical examples, first.

Back to Geometry

The notion of a full subset can be used to interpret a sentence like "for practically all points P, Q: P is distinct from Q", by using the product topology on $E \times E$ (where E denotes the Euclidean plane). It is easy to see that this sentence, thus interpreted, is true: $\{(P, Q) \in E \times E \mid P \neq Q\}$ is full in $E \times E$, since it is both open and dense in $E \times E$. The statement "for practically all points, P, and practically all lines, ℓ, : P is not on ℓ" requires the definition of a topology on the collection of lines. We will use the construction explained in appendix B, using the idea of a quotient topology. In the sequel, G denotes the collection of lines in the Euclidean plane (while E denotes the collection of points of the Euclidean plane).

14.1 Proposition $E \times G \to \{\ (P, \ell) \in E \times G \mid P \text{ is not on } \ell\ \}$.

Proof: Define $H := \{\ (x, y, p, q, r) \in \mathbb{R}^5 \mid (p, q) = (0, 0)\ \}$.

Then $\mathbb{R}^5 \setminus H$ equals $E \times (\mathbb{R}^3 \setminus \{(0, 0)\} \times \mathbb{R})$.

According to Appendix B (in particular, the subsection about the topology on the set of lines in the Euclidean plane), G can be identified with a quotient space of the space $\mathbb{R}^3 \setminus (\{(0, 0)\} \times \mathbb{R})$. As a consequence, there exists an equivalence relation $\sim$ on $\mathbb{R}^5 \setminus H$ and a quotient map

$\pi\colon \mathbb{R}^5 \setminus H \to (\mathbb{R}^5 \setminus H) / \sim$

such that

i) $(\mathbb{R}^5 \setminus H) / \sim$ can be identified with $E \times G$,

ii) $\{ (P, \ell) \in E \times G \mid P \text{ is not on } \ell \}$ can be identified with a set

$a \subseteq (\mathbb{R}^5 \setminus H) / \sim$ such that

$\pi^{-1}(a) = \{ (x, y, p, q, r) \in \mathbb{R}^5 \setminus H \mid px + qy + r \neq 0 \}$.

But the set $\{ (x, y, p, q, r) \in \mathbb{R}^5 \mid px + qy + r \neq 0 \}$ is full in $\mathbb{R}^5$, since it is open and dense in $\mathbb{R}^5$. Hence, it is full in $\mathbb{R}^5 \setminus H$, by Proposition 5.11 i). The proposition now follows from Lemma 14.2 below.

□

14.2 Lemma Let X be a topological space, $\sim$ an equivalence relation on X such that the quotient map $\pi\colon X \to X/\sim$ is an open map.

Then for every $a \subseteq X$:

a is full in $X/\sim$ (only) if $\pi^{-1}(a)$ is full in X.

Proof :

Suppose that a is full in $X/\sim$.

Then for every nonempty O, open in X,

$\pi(O)$ is nonempty and open in $X/\sim$ (since π is open).

Since a is full in $X/\sim$, there is a nonempty $O' \subseteq \pi(O)$, open in $X/\sim$, such that $O' \subseteq a$.

Then $O \cap \pi^{-1}(O')$ is nonempty and open in X, and

$O \cap \pi^{-1}(O') \subseteq \pi^{-1}(a)$.

Hence, $\pi^{-1}(a)$ is full in X.

On the other hand, suppose that $\pi^{-1}(a)$ is full in X.

Then for every nonempty O, open in $X/\sim$,

the set $\pi^{-1}(O)$ is nonempty and open in X.

Hence, there is an $O' \subseteq \pi^{-1}(O)$ such that

$O' \neq \varnothing$ and $O' \subseteq \pi^{-1}(a)$.

Then $\pi(O') \subseteq a$.

Since π is open, $\pi(O')$ is open in $X/\sim$.

And $\pi(O') \neq \varnothing$, since $O' \neq \varnothing$.

Hence, a is full in $X/\sim$.

□

Likewise, it is possible to give a precise meaning to sentences like : "given a line, practically all lines are not perpendicular to that line", "for practically all pairs (m, ℓ) of lines : ℓ and m are not parallel", "for practically all pairs (m, ℓ) of lines : ℓ and m are distinct", etc., etc.

Some Warnings

In classical logic, universal quantification has a number of properties that allow us to take a certain amount of notational freedom. For example,

$\forall x \in X$ [$\forall y \in Y$ [a(x,y)]]

is essentially the same statement as

$\forall y \in Y$ [$\forall x \in X$ [a(x,y)]]

or $\forall x \in X, y \in Y$ [a(x, y)].

As we will prove, below, our non-standard universal quantification does not allow this freedom, and we will have to be very careful and specific when formulating such statements.

Another point of deviation, also having notational consequences, is the following. In classical logic, if $a, b \subseteq X$, then the statement

$\forall x \in a$ [$x \in b$]

is equivalent to

$\forall x \in X$ [$x \notin a$ or $x \in b$].

This allows us to simplify formal language considerably, by using notations like $\forall x$ [$x \notin a$ or $x \in b$].

However,

"for practically all $x \in a$: $x \in b$" (i.e., $a \to b$)

is not equivalent to

"for practically all $x \in X$: $x \notin a$ or $x \in b$" (i.e., $X \to a^c \cup b$)

(see 5.12 iv)

nor to "for practically all $x \in X$: $x \in O(a, b)$" (i.e., $X \to O(a, b)$)

(see Example 9.7 : $O(\ell, \ell^c) = \ell^c$, but $\ell \to \ell^c$ is not true).

It is not clear whether these phenomena are only annoying coincidences of the mathematical elaborations chosen, or that they are intrinsic to the intuitive notions we tried to capture. Concerning the "non-commutativity" of the quantifiers [Weydert 93] found similar results in a probabilistic approach. On the other hand, our counterexamples, as used in the proofs below, are of a highly artificial nature. Moreover, in 14.9 below, we will show that there exist other elaborations in which the quantifiers do commute.

14.3 Proposition / Warning

Let X and Y be topological spaces, and $a \subseteq X \times Y$.

"For practically all $x \in X$: for practically all $y \in Y$: $(x, y) \in a$"

does not imply

"For practically all $y \in Y$: for practically all $x \in X$: $(x, y) \in a$".

Proof / counterexample :
Define $X = Y := \mathbb{Q}$ and let φ be a bijection $\mathbb{Q} \to \mathbb{N}$.

Define $v := \{(x, y) \in X \times Y \mid \varphi(y) - \frac{1}{2} < x < \varphi(y) + \frac{1}{2}\}$
and $a := v^c$ (that is, $(X \times Y) \setminus v$).
Then, for all $x \in X$,
the set $\{ y \in Y \mid (x, y) \in a \}$ is full in Y,
since $\{ y \in Y \mid (x, y) \in a^c \}$ contains at most one element.
Hence,

for practically all $x \in X$: for practically all $y \in Y$: $(x, y) \in a$

is true.
But, for every $y \in Y$,
the set $\{ x \in X \mid (x, y) \in a \}$ is not full in X,
since $\{ x \in X \mid (x, y) \in a^c \}$ contains a non-empty open interval.
Hence,

for practically all $y \in Y$: for practically all $x \in X$: $(x, y) \in a$

is not true.
□

14.4 Corollary / Warning
Let X be a topological space, and $a \subseteq X$.

"For practically all $(x, y) \in X \times Y$: $(x, y) \in a$"

does not imply

"For practically all $y \in Y$: for practically all $x \in X$: $(x, y) \in a$".

Proof: With X, Y and a as in the proof of 14.3, it is not difficult to see that $X \times Y \to a$ is true.
□

14.5 Proposition / Warning
Let X and Y be topological spaces, and $a \subseteq X \times Y$.

"For practically all $x \in X$: for practically all $y \in Y$: $(x, y) \in a$"

does not imply

"For practically all $(x, y) \in X \times Y$: $(x, y) \in a$".

Proof / counterexample :
Define $X = Y := \mathbb{R}$. A *rational block* is a nonempty open subset of $\mathbb{R} \times \mathbb{R}$ of the form

$\{ (p, q) \in \mathbb{R} \times \mathbb{R} \mid \alpha < p < \beta \text{ and } \gamma < q < \delta \}$

for some $\alpha, \beta, \gamma, \delta \in \mathbb{Q}$ (such that $\alpha < \beta$ and $\gamma < \delta$).
Let $O_1, O_2, \ldots$ be an enumeration of all rational blocks. Construct a row

$(x_1, y_1), (x_2, y_2), \ldots$

of elements of $\mathbb{R} \times \mathbb{R}$ such that (for i, j = 1, 2, ...)

$(x_i, y_i) \in O_i$

and $y_i \neq y_j$ whenever $i \neq j$.

Define $v := \{ (x_1, y_1), (x_2, y_2), \ldots \}$ and $a := v^c$.

Then, for every $x \in X$,

the set $\{ y \in Y \mid (x, y) \in a^c \}$ contains at most one element, hence

$Y \rightarrow \{ y \in Y \mid (x, y) \in a \}$.

Hence,

for practically all $x \in X$: for practically all $y \in Y$: $(x, y) \in a$

is true.

But $v = a^c$ is dense in $X \times Y$, hence $X \times Y \rightarrow a$ is not true.

Hence,

for practically all $(x, y) \in X \times Y$: $(x, y) \in a$

is not true.

□

Independence

Now that we have seen some properties that non-strict universal quantification does not satisfy, we will present one property that it does satisfy and that makes it suitable for the purposes of Chapter 5.

14.6 Theorem (Independence Theorem)

Let X and Y be topological spaces and $a, b \subseteq X$.

Then $a \rightarrow b$ implies $a \times Y \rightarrow b \times Y$.

(In other words, $a(x) \rightarrow b(x)$ implies $a(x) \wedge c(y) \rightarrow b(x)$.)

Proof: Although this result is an easy corollary of Lemma 14.2, we will give an elementary proof.

Suppose that b is full in a.

Then, for every O, open in $X \times Y$, such that $O \cap (a \times Y) \neq \emptyset$,

there are O_X, O_Y, open in X and Y, respectively, such that

$O_X \times O_Y \subseteq O$

and $(O_X \times O_Y) \cap (a \times Y) \neq \emptyset$.

Then $O_X \cap a \neq \emptyset$.

Since b is full in a, there is an $O' \subseteq O_X$ such that

$O' \cap a \neq \emptyset$ and $O' \cap a \subseteq b$.

Then $(O' \times O_Y) \cap (a \times Y) \neq \emptyset$, and

$(O' \times O_Y) \cap (a \times Y) \subseteq b \times Y$.

Hence, $a \times Y \to b \times Y$.

□

It is also true that $a \times Y \to b \times Y$ implies $a \to b$, provided that $Y \neq \emptyset$. This will be referred to as *the reverse of the independence theorem.*

14.7 Corollary Let X be a topological space and $a, b \subseteq X$.
Then $a \to b$ implies $(a \cap b^c) \times a \to X \times b$.
In other words, $a(x) \to b(x)$ implies $(a(s) \wedge \neg b(s) \wedge a(t)) \to b(t)$.
(Proof: By 14.6, $a \to b$ implies $(a \cap b^c) \times a \to (a \cap b^c) \times b$, which implies $(a \cap b^c) \times a \to X \times b$, by 5.10 iv.)

This result states that our definitions (and, essentially, we are still investigating Definition 5.7) display behaviour that is typically associated with rules-with-possible-exceptions. If $a(x) \to b(x)$ is valid-as-a-rule, then the occurrence of an exception does not invalidate this rule, and one is inclined to apply it again, on a next occasion: the premisses $a(s)$, $\neg b(s)$ and $a(t)$ allow $b(t)$ as a conclusion. Of course, $a(s)$, $\neg b(s)$, $a(t)$ and $s = t$ do no longer allow $b(t)$ as a conclusion.
In short : **if a rule is valid but has an exception, the rule still applies in other, "independent" situations.**
This seems to be an essential general principle of the calculus of rules-with-possible-exceptions.

With our definitions, the principle above only applies if the new situation is entirely independent from the former one. It does not apply, in general, if this independence is only partial. For example, if $a(s, t)$ and $b(s, t)$ are predicates of two variables, then the fact that $a(s, t) \to b(s, t)$ is valid-as-a-rule does not imply that $a(s, t) \wedge \neg b(s, t) \wedge a(s', t) \to b(s', t)$ is valid-as-a-rule, not even if b does not rely on t :

14.8 Proposition / Warning
Let X and Y be topological spaces, $a \subseteq X \times Y$ and $b \subseteq X$, such that

$$a \to b \times Y.$$

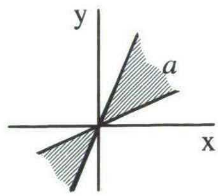
Then, in general, it is not true that
$\{ (x_1, x_2, y) \in X \times X \times Y \mid (x_1, y) \in a, (x_2, y) \in a \text{ and } x_1 \notin b \} \to$
$\{ (x_1, x_2, y) \in X \times X \times Y \mid x_2 \in b \}$
(In other words :
$a(x, y) \to b(x)$ does not imply $a(x_1, y) \wedge \neg b(x_1) \wedge a(x_2, y) \to b(x_2)$.)

Proof / counterexample :
Define $X = Y := \mathrm{IR}$,

$$a(x, y) :\Leftrightarrow \tfrac{1}{2}x \le y \le 2x \qquad \text{(and } a := \{ (x, y) \in X \times Y \mid a(x, y) \}\text{)}$$

$$b(x) :\Leftrightarrow x \neq 0 \qquad \text{(and } b := \{ x \in X \mid b(x) \}\text{)}$$

Then $a \to b \times Y$ is true.



But $a(x_1, y), \neg b(x_1)$ implies $x_1 = y = 0$,
and $y = 0$, $a(x_2, y)$ implies $x_2 = 0$, i.e., $\neg b(x_2)$.
Hence, $a(x_1, y) \wedge \neg b(x_1) \wedge a(x_2, y) \to \neg b(x_2)$ is true.
Since the combination of assumptions $a(x_1, y), \neg b(x_1), a(x_2, y)$ is consistent (being equivalent to $x_1 = x_2 = y = 0$),

$$a(x_1, y) \wedge \neg b(x_1) \wedge a(x_2, y) \to b(x_2)$$

is not true.

□

Nested quantificational clauses

Let X and Y be topological spaces, and $a \subseteq X \times Y$. The "proposition"

"for practically all $y \in Y$: $(x, y) \in a$"

is a statement about $x \in X$. Hence, its extension is some subset of X. In 14.3, 14.4 and 14.5, we assumed without further thought that this extension equals $\{ x \in X \mid \{x\} \times Y \to a \}$. But this is not the only possibility. For example, inspired by the definition of O(,), we could define the extension to be

$$\bigcup_{\substack{O \subseteq X, \\ O \times Y \to a}} O$$

(That is, the largest $O \subseteq X$ such that $O \times Y \to a$.)

To get an idea of the difference: if $X = Y = \mathrm{IR}$, then according to the original, straightforward definition, 0 is in the extension of

"for practically all $y \in Y$: $x = 0$"

but not in the extension of

"for practically all $y \in Y$: $x \neq 0$".

According to the new definition, however, 0 *is* in the extension of

"for practically all $y \in Y$: $x \neq 0$"

but not in the extension of

"for practically all $y \in Y$: $x = 0$".

Proof: For all $O \subseteq X$: $O \to \{0\}^c$, hence $O \times Y \to \{0\}^c \times Y$.

Hence, if $a = \{0\}^c \times Y$, then

$$0 \in \bigcup_{\substack{O \subseteq X, \\ O \times Y \to a}} O$$

On the other hand, there is no $O \subseteq X$ such that $O \times Y \to \{0\} \times Y$ and $0 \in O$. Hence, if $a = \{0\} \times Y$, then

$$0 \notin \bigcup_{\substack{O \subseteq X, \\ O \times Y \to a}} O$$

$\square$

Although this may seem awkward, at first sight, it becomes quite natural if we realize that this construction describes in which *regions* (= open sets) a quantificational statement should be considered true, while the straightforward construction is concerned about *points*.

Using this construction (or any other alternative) will also have notational consequences. For example, if X and Y are topological spaces, a(x, y) is a predicate with extension $a \subseteq X \times Y$ and $x_0 \in X$, then

"for practically all $y \in Y$: $a(x_0, y)$"

could denote

$Y \to \{ y \in Y \mid (x_0, y) \in a \}$

(that is equivalent to $\{x_0\} \times Y \to a$), but also

$$x_0 \in \bigcup_{\substack{O \subseteq X, \\ O \times Y \to a}} O$$

(that is equivalent to: for some (open) $O \subseteq X$ containing x_0, $O \times Y \to a$).

This notational confusion is a serious disadvantage. The construction above, however, also has one major advantage: using this construction to interpret nested quantificational sentences yields commutating quantifiers, as we will see, below.

In the theorem below, $a \subseteq X \times Y \times Z$ is the extension of some ternary predicate a(x, y, z), O_x denotes the extension of

"for practically all $x \in X$: a(x, y, z)",

$O_{x,y}$ denotes the extension of

"for practically all $y \in Y$: for practically all $x \in X$: a(x, y, z)".

Likewise for O_y and $O_{y,x}$.

O_z denotes the extension of

"for practically all $(x, y) \in X \times Y$: a(x, y, z)".

Then the theorem states that, with the new definition of these extensions, the three sentences

"for practically all $x \in X$: for practically all $y \in Y$: a(x, y, z)",

"for practically all $y \in Y$: for practically all $x \in X$: a(x, y, z)",

and "for practically all $(x, y) \in X \times Y$: a(x, y, z)"

are mutually equivalent (that is, they have the same extensions).

14.9 Theorem Let X, Y and Z be topological spaces, and $a \subseteq X \times Y \times Z$.
Let O_x be the largest open subset of $Y \times Z$ such that $X \times O_x \to a$.
Let $O_{x,y}$ be the largest open subset of Z such that $Y \times O_{x,y} \to O_x$.

Let O_y be the largest open subset of $X \times Z$ such that $Y \times O_y \to a$.
Let $O_{y,x}$ be the largest open subset of Z such that $X \times O_{y,x} \to O_y$.

Let O_z be the largest open subset of Z such that $X \times Y \times O_z \to a$.
Then

$$O_{x,y} = O_z = O_{y,x}.$$

Proof:
1) $X \times Y \times O_z \to a$, hence $Y \times O_z \subseteq O_x$, hence, $O_z \subseteq O_{x,y}$.
2) $Y \times O_{x,y} \to O_x$, hence (by Theorem 14.6),

$$X \times Y \times O_{x,y} \to X \times O_x.$$

But $X \times O_x \to a$. Hence, by the open-lemma and 5.11, $X \times Y \times O_{x,y} \to a$.
Hence, $O_{x,y} \subseteq O_z$.
Hence, $O_{x,y} = O_z$. Likewise, $O_{y,x} = O_z$.
□

In §12, we saw that even within the sphere of influence of Definition 5.7, there is a variety of plausible definitions for (the extension of) implicational statements. Likewise, there does not exist a unique, most natural definition of the extension of quantificational statements. As long as we restrict ourselves to non-nested quantificational clauses, however, there is no need for a separate definition of an extension (in view of Definition 5.7). Hence, for the rest of this thesis, we will adhere to the straightforward interpretation of quantificational-statements-with-possible-exceptions, despite possible advantages (see, for example, Theorem 14.9) of other constructions.

§15 Examples

As our original aim was the study of practical reasoning, we will discuss, in this section, a number of situations from practical reasoning in which non-strict universal quantification plays a role.

Crowds

With the definitions of §14, it is easy to see that

for practically all $(x, y) \in \mathbb{R} \times \mathbb{R}$: $x \neq y$.

This phenomenon is not unusual in practical reasoning. The sentence expresses that, given elements x and y, we may safely conclude that x is distinct from y, disregarding exceptions, which seems reasonable for extremely large and homogeneous sets, such as the set of all birds, the set of all human beings, or even the set of spectators in a crowded footballstadion.

15.1 Definition A topological space X is called a *crowd* whenever

for practically all $(x, y) \in X \times X$: $x \neq y$.

For example, $\mathbb{R}$ and the Euclidean plane are crowds, finite spaces are not.

15.2 Proposition A topological space X is a crowd (only) if every nonempty O (open in X) contains nonempty O' and O'' such that

$$O' \cap O'' = \varnothing.$$

Proof: Suppose that X is a crowd, that is

$\{ (x, y) \in X \times X \mid x \neq y \}$ is full in $X \times X$.

If O is nonempty and open in X, then $O \times O$ is nonempty,

hence there are O' and O'' such that

$$O' \times O'' \subseteq O \times O,$$
$$O' \times O'' \neq \varnothing,$$
$$O' \times O'' \subseteq \{ (x, y) \in X \times X \mid x \neq y \}.$$

Hence, $O' \subseteq O$, $O'' \subseteq O$, $O' \neq \varnothing$, $O'' \neq \varnothing$ and $O' \cap O'' = \varnothing$.

On the other hand, suppose that every nonempty O contains nonempty O' and O'' such that $O' \cap O'' = \varnothing$.

Let O be nonempty and open in $X \times X$. Say,

$O_1 \neq \varnothing$, $O_2 \neq \varnothing$ and $O_1 \times O_2 \subseteq O$.

Then either $O_1 \times O_2 \subseteq \{ (x, y) \in X \times X \mid x \neq y \}$ (and there is nothing left to be proved), or

$$O_1 \cap O_2 \neq \varnothing.$$

In that case, there are nonempty O' and O'' such that

$$O' \subseteq O_1 \cap O_2 , O'' \subseteq O_1 \cap O_2 \text{ and } O' \cap O'' = \varnothing$$

(because of the assumption).

Then $O' \times O'' \subseteq O$, $O' \times O'' \neq \varnothing$, and

$$O' \times O'' \subseteq \{ (x, y) \in X \times X \mid x \neq y \}.$$

Hence, X is a crowd.

□

15.3 Corollary If X is a crowd, then X is infinite.

15.4 Corollary If X is a Hausdorff space without isolated points (see B.2 and B.9), then X is a crowd.

Proof: Suppose that X is a Hausdorff space without isolated points. Then every nonempty $O \subseteq X$ contains at least two distinct points, hence it contains two disjoint nonempty open subsets. By 15.2, X is a crowd.

□

In a crowd every element is exceptional :

15.5 Proposition Let X be a crowd, and $p \in X$. Then $X \setminus \{p\}$ is full in X.

Proof: Let $O \subseteq X$ be nonempty.

By 15.2, there are nonempty $O_1, O_2 \subseteq O$ such that $O_1 \cap O_2 = \varnothing$.

Then either $p \notin O_1$ or $p \notin O_2$.

Hence, O contains a nonempty open subset of $X \setminus \{p\}$.

Hence $X \to X \setminus \{p\}$.

□

15.6 Example Let X be a crowd, and $x_0 \in X$. Let $R \subseteq X \times X$ be the relation defined by :

$$x \, R \, y \;:\Leftrightarrow\; x \neq x_0 \text{ and } y \neq x_0.$$

Then for practically all $(x, y) \in X \times X$: $x \, R \, y$,

(and even for practically all $x \in X$: for practically all $y \in Y$: $x \, R \, y$),

but not for practically all $y \in Y$: $x_0 \, R \, y$.

Reading x R y as "person x knows person y", then the situation is one in which (practically) every person in the crowd (X) knows everyone, but there is one person (x_0) in the crowd that does not know anyone, and is not known by anyone.

The Paradox of the Barber

In a town there is a (male) barber, who is said to *shave those and only those men in town who do not shave themselves*. Symbolically represented, $\mathtt{b}$ is such that

$$\left.\begin{array}{ll} & \mathtt{shaves}(\mathtt{b}, y) \rightarrow \neg\mathtt{shaves}(y, y) \\ \text{and} & \neg\mathtt{shaves}(y, y) \rightarrow \mathtt{shaves}(\mathtt{b}, y). \end{array}\right\} \quad (*)$$

According to classical logic, $(*)$ implies

$$\mathtt{shaves}(\mathtt{b}, \mathtt{b}) \leftrightarrow \neg\mathtt{shaves}(\mathtt{b}, \mathtt{b}),$$

a contradiction.

In practice, however, if a general statement "obviously" amounts to a contradiction when applied to a particular instance, then it is equally obvious that that general statement was meant to exclude that particular instance. For example, if John says to Mary : "All the girls I have seen were considerably less beautiful than you," this does not apply to Mary herself, as it is obvious that Mary is not less beautiful than herself. In fact, it is not unusual for a general statement to have "obvious" exceptions for other but similar reasons. For example, if John says to Mary: "I have never seen a girl as beautiful as you," this does not imply that John never saw Mary, although Mary is as beautiful as herself.

Likewise, the sentence "the barber shaves those and only those men in town who do not shave themselves," interpreted as a sentence that could occur in everyday life, rather than in a logic textbook, does not imply "the barber shaves himself if and only if he does not shave himself."

Indeed, if we use our non-standard interpretation, then $(*)$ is not contradictory :

15.7 Example For $x, y \in \mathbb{R}$, define

$$\mathtt{S}(x, y) :\Leftrightarrow \ (x = y \text{ and } y \in \mathbb{Q}) \text{ or } (x = 0 \text{ and } y \notin \mathbb{Q}).$$

Then $\{ y \in \mathbb{R} \mid \mathtt{S}(0, y) \} = \mathbb{R} \setminus \mathbb{Q} \cup \{0\}$, and

$$\{ y \in \mathbb{R} \mid \neg\mathtt{S}(y, y) \} = \mathbb{R} \setminus \mathbb{Q} .$$

Hence, $\{ y \in X \mid \mathrm{S}(0, y) \} \leftrightarrow \{ y \in X \mid \neg\mathrm{S}(y, y) \}$.

What is a Constant ?

In the classical example of a logical argument :

$$\frac{\begin{array}{l}\text{Every human is mortal,}\\ \text{Socrates is human}\end{array}}{\text{Socrates is mortal}}$$

Socrates is a constant. This essentially means that "to be Socrates" is (assumed to be) a predicate that applies to one and only one object. That is,

if x "is Socrates" and y "is Socrates", then x equals y.

Let us, therefore, define :

15.8 Definition If X is topological space, then a constant (in X) is a $C \subseteq X$ such that

i) $C \neq \varnothing$,

ii) $C \times C \rightarrow \{ (x, y) \in C \times C \mid x = y \}$
(in short : $x \in C, y \in C \rightarrow x = y$).

15.9 Example If X is a topological space, and $p \in X$, then $\{p\}$ is a constant. $\{0, 1\}$ is not a constant in $\mathbb{R}$.

15.10 Example If $X = \{1, 2\}$, $\tau = \{\varnothing, \{1\}, X\}$, $C = X$ is a constant. (Note that the set $\{(1,1)\}$ is the smallest nonempty open subset of $C \times C$ with product topology. Hence, a subset $a \subseteq C \times C$ is full in $C \times C$ (only) if $(1,1) \in a$.)

A constant in X that contains more than one element could be thought of as a *vague element* of X. Note that "Socrates has property P" just means that being Socrates implies having property P. Likewise, if a(x) is a predicate with extension $a \subseteq X$, and C is a constant in X, then "C has property a" (that is, "a(C)") just means

$x \in C \rightarrow x \in a$.

For example, if X and C are as in Example 15.10, and $a = \{1\}$, then a(C) is true, but

$x \in C, x \notin a \nrightarrow \perp$

(since $C \setminus a = \{2\}$, and $\{2\} \not\twoheadrightarrow \varnothing$).

Hence, if a vague element has a property, the denial of this property (for the same vague element) is not necessarily inconsistent. Of course, for ordinary elements this cannot occur.

15.11 Proposition C is a constant (only) if there is a $p \in C$ such that

i) $\{p\}$ is open in C,

ii) for all nonempty O, open in C : $p \in O$.

Proof: 1) Suppose that C is a constant. Let O be a nonempty open subset of C-with-induced-topology. Then $O \times O$ is nonempty and open in $C \times C$. Since C is a constant, there exist O_1, O_2, open in C, such that

$$O_1 \times O_2 \subseteq O \times O,$$
$$O_1 \times O_2 \neq \varnothing, \quad \text{(hence, } O_1 \neq \varnothing \text{ and } O_2 \neq \varnothing\text{)}$$
$$O_1 \times O_2 \subseteq \{ (x, y) \in X \times X \mid x = y \}.$$

Hence, there is an element $p \in C$ such that $O_1 = O_2 = \{p\}$.

In particular, $\{p\}$ is open in C.

Now suppose that there exists a nonempty O', open in C, such that $p \notin O'$.

Then $C \times C \not\twoheadrightarrow \{ (x, y) \in C \times C \mid x = y \}$.

(The latter isn't even dense in the former, since $O' \times \{p\}$ is nonempty and open in $C \times C$, but does not contain elements of $\{ (x, y) \in C \times C \mid x = y \}$.)

But C is a constant. Contradiction.

Hence, every nonempty open subset of C contains p.

2) On the other hand, suppose that $p \in C$ is an element satisfying i) and ii).

Then every nonempty open subset of $C \times C$ contains $\{p\} \times \{p\}$.

Hence, $\{p\} \times \{p\}$ is full in $C \times C$.

Hence, $\{ (x, y) \in C \times C \mid x = y \}$ is full in $C \times C$, by 5.10 iv).

Moreover, $C \neq \varnothing$, since $p \in C$.

Hence, C is a constant.

□

15.12 Corollary If X is a Hausdorff space, and C is a constant, then there is a $p \in C$ such that $C = \{p\}$.

Proof: Suppose that C is a constant.

Let p be the (unique) element of C satisfying 15.11 i) and ii).

Then p is the only element of C such that $\{p\}$ is open in C.

If X is a Hausdorff space, then so is C.

Hence, if $C \neq \{p\}$, say $q \in C$ and $q \neq p$, then there exist O_p, O_q, open in C, such that

$$p \in O_p,\ q \in O_q \text{ and } O_p \cap O_q = \varnothing.$$

In particular, O_q is nonempty and $p \notin O_q$. Contradiction with 15.11 ii).
Hence, $C = \{p\}$.

□

The Sorites Paradox

Consider the following slight adaptation of the Sorites paradox (see §3 or [Hjelmslev 23]). The combination of assumptions

i) If I can lift n milligrams, then I can lift $n + 1$ milligrams
ii) I can lift 2 milligrams

does not imply

iii) I can lift any weight.

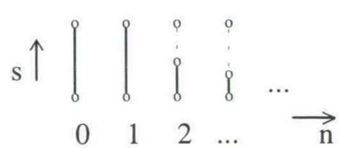
Hence, in some way, the (vague) "set" $\{ n \in \mathbb{N} \mid \text{I can lift } n \text{ milligrams} \}$ is a counterexample to the principle of induction. The crucial point seems to be that the implicational statement i) is valid only as-a-rule, while at the same time, no concrete exceptional $n \in \mathbb{N}$ can be indicated. Although it is questionable whether the defeasible conditional at issue here is not entirely different from the defeasible conditional studied in this thesis, it is nevertheless possible to represent the Sorites paradox using our topological notions :

On the interval (0, 1) ($:= \{x \in \mathbb{R} \mid 0 < x < 1\}$) we define the following topology :

$$\tau := \{ (0, x) \mid x \in [0, 1] \}.$$

It is easy to check that ((0, 1), τ) is a topological space, be it a somewhat unusual (pathological) one. To distinguish this space from the interval (0, 1) with the topology induced from $\mathbb{R}$, we denote ((0, 1), τ) by $\mathcal{S}$.

Define $V := \{ (n, s) \in \mathbb{N} \times \mathcal{S} \mid s < \frac{1}{n} \}$



We intend to think of $\mathcal{S}$ as a collection of possible worlds, each containing the objects 0, 1, 2, ... Then V could be said to represent a (vague) sub"set"

$\mathcal{V}$ of $\mathbb{N}$: instead of $(n, s) \in V$, we will say "in world s, n is an element of $\mathcal{V}$", or write $n \in_s \mathcal{V}$.

This "set" $\mathcal{V}$ fails to satisfy, in some way, the principle of induction.
Since $\{ s \in \mathcal{S} \mid n \in_s \mathcal{V} \} \to \{ s \in \mathcal{S} \mid n + 1 \in_s \mathcal{V} \}$ (as is not difficult to prove), we could say that the "set" $\mathcal{V}$ satisfies the sentence

if n is an element of $\mathcal{V}$, then n + 1 is an element of $\mathcal{V}$,

interpreted as a rule-with-possible-exceptions.
Since $\{ s \in \mathcal{S} \mid 0 \in_s \mathcal{V} \} = \mathcal{S}$, the "set" $\mathcal{V}$ satisfies

0 is an element of $\mathcal{V}$.

But, for example, $\{ s \in \mathcal{S} \mid 37 \notin_s \mathcal{V} \} \not\to \varnothing$.
Moreover, $\{ s \in \mathcal{S} \mid \text{for all } n \in \mathbb{N} : n \in_s \mathcal{V} \} = \varnothing$.

In this sense, $\mathcal{V}$ models sets like $\{ n \in \mathbb{N} \mid$ I can lift n milligrams $\}$.

Equivalence Relations

As is well known, an equivalence relation on a set X is a binary relation on X that is symmetric, transitive and reflexive, i.e., a relation $\sim$ satisfying

for all $x, y \in X$ such that $x \sim y$: $y \sim x$	(symm)
for all $x, y, z \in X$ such that $x \sim y$ and $y \sim z$: $x \sim z$	(trans)
for all $x \in X$: $x \sim x$	(refl)

If X is equipped with a topology, it makes sense to talk about relations satisfying, for example, (symm) up-to-possible-exceptions. It is not difficult to give an example of a space and a relation satisfying (symm), (trans) and (refl) thus interpreted, such that each of them has exceptions :
For $p, q \in \mathbb{R}$, define

$$p \sim q \ := \ p - q \in \mathbb{Z} \ , (p, q) \neq (0, 0) \text{ and } (p, q) \neq (1, 0).$$

Then

$$\{ (x, y) \in \mathbb{R} \times \mathbb{R} \mid x \sim y \} \to \{ (x, y) \in \mathbb{R} \times \mathbb{R} \mid y \sim x \}$$

(in short : $x \sim y \to y \sim x$).
Hence, (symm) is valid-up-to-possible-exceptions. (Symm) indeed has exceptions : $0 \sim 1$, but $1 \nsim 0$.
Likewise, (trans) is valid, up to exceptions : $1 \sim 2, 2 \sim 0$, but $1 \nsim 0$.
(refl) is also valid-up-to-possible-exceptions, since

$$\mathbb{R} \to \{ x \in \mathbb{R} \mid x \sim x \}.$$

However, $0 \not\sim 0$.
In this way, the relation ~ on $\mathbb{R}$ satisfies-up-to-possible-exceptions the requirements for an equivalence relation, although it is not symmetric, not transitive, and not reflexive.

Preferential Consequence Relations
Something similar can be done with preferential consequence relations instead of equivalence relations :

15.13 Proposition There exists a triple $(B, \leq, \mathrel{|\!\sim})$ such that
i) $(B, \leq)$ is a Boolean algebra equipped with a topology,
ii) $\mathrel{|\!\sim}$ is a binary relation on B,
iii) B is a crowd (see Definition 15.1),
iv) P1-P5 are valid-up-to-possible-exceptions, that is:
for practically all a ($\in$ B) : $a \mathrel{|\!\sim} a$,
for practically all a, b, c such that $a \mathrel{|\!\sim} b$ and $b \leq c$: $a \mathrel{|\!\sim} c$,
for practically all a, b, c such that $a \mathrel{|\!\sim} b$ and $a \wedge b \mathrel{|\!\sim} c$: $a \mathrel{|\!\sim} c$,
for practically all a, b, c such that $a \mathrel{|\!\sim} b$ and $a \mathrel{|\!\sim} c$: $a \wedge b \mathrel{|\!\sim} c$,
for practically all a, b, c such that $a \mathrel{|\!\sim} c$ and $b \mathrel{|\!\sim} c$: $a \vee b \mathrel{|\!\sim} c$.
v) The rule of monotony is valid-up-to-possible-exceptions
(that is, for practically all a, b, c such that $a \mathrel{|\!\sim} b$: $a \wedge c \mathrel{|\!\sim} b$,
vi) The rule "if $a \mathrel{|\!\sim} b$, then not $a \mathrel{|\!\sim} \neg b$" is valid-up-to-possible-exceptions. That is,
$$\{ (a, b) \in B \times B \mid a \mathrel{|\!\sim} b \} \to \{ (a, b) \in B \times B \mid a \mathrel{|\!\not\sim} \neg b \}.$$
Proof: Let B be the Boolean algebra $(\mathcal{P}(\mathbb{N}), \subseteq, \cap, \cup, .^c, \emptyset, \mathbb{N})$.
The topology on B is defined as follows :
$V \subseteq \mathcal{P}(\mathbb{N})$ is open whenever for all $a \in V$ there is an $n \in \mathbb{N}$ such that
$$O_{a,n} := \{ b \in \mathcal{P}(\mathbb{N}) \mid b \cap \{1, \ldots, n\} = a \cap \{1, \ldots, n\} \} \subseteq V.$$
With this topology, $\mathcal{P}(\mathbb{N})$ is a Hausdorff space without isolated points, as is easy to see. By Corollary 15.4, B is a crowd.
For $a, b \in B$, define
$$a \mathrel{|\!\sim} b \;:\Leftrightarrow\; a \subseteq b \text{ and } (a, b) \neq \mathcal{E},$$
where $\mathcal{E} = \{ (\{1\}, \{1\}),\ (\{1,2\}, \{1,2,3\}) \}$.
Then it is straightforward (but tedious) to prove that $(B, \mathrel{|\!\sim})$ satisfies iv), v) and vi) of the proposition.

Although some of the rules do not have exceptions this time, a considerable number of them do. P1 has exceptions, since $\{1\} \not\mathrel{|\!\sim} \{1\}$.

P2 has exceptions, since

$$\{1,2\} \mathrel{|\!\sim} \{1,2\} \text{ and } \{1,2\} \subseteq \{1,2,3\} \text{ but } \{1,2\} \not\mathrel{|\!\sim} \{1,2,3\}.$$

P3 and P4 do not have exceptions.

P5 has exceptions, since

$$\{1\} \mathrel{|\!\sim} \{1,2,3\} \text{ and } \{2\} \mathrel{|\!\sim} \{1,2,3\}, \text{ but } \{1,2\} \not\mathrel{|\!\sim} \{1,2,3\}.$$

The rule of monotony has exceptions, since

$$\{1,2,3\} \mathrel{|\!\sim} \{1,2,3\}, \text{ but } \{1,2\} \not\mathrel{|\!\sim} \{1,2,3\}.$$

The rule mentioned in vi) also has exceptions, since

$$\varnothing \mathrel{|\!\sim} \varnothing, \text{ but } \varnothing \not\mathrel{|\!\sim} \mathbb{N} \text{ is not true.}$$

□

This proposition shows how our non-standard interpretation of universal quantification can be used to *enlarge the expressive power of the language used to formulate plausibility postulates for reasoning behaviour.*

A triple (B, $\leq$, $\mathrel{|\!\sim}$) that satisfies the requirements of the proposition can be seen as a caricature of a person that accepts-up-to-possible-exceptions the rules of classical propositional logic. This caricature is better than the one in Chapter 3 in at least one respect: the interpretation of validity-up-to-possible-exceptions is more direct and natural. Note that by requirement iii) and Proposition 15.5, every single sentence of B is exceptional. This effect could never have been achieved with preferential semantics. However, the system above is not very useful, in that almost all combinations of postulates will have a representation of this kind. Moreover, there is no nesting of implicational statements. That is, unlike the system of Chapter 3, the idea of Proposition 15.13 does not give rise to a (useful) notion of validity, like topological validity. (This is the main reason why we did not elaborate the tedious parts of the proof of 15.13.) The question remains, how to use our findings in a more productive way. This will be the subject of the next chapter.

Chapter 5
Possible Meanings Semantics

The topological notions of Chapter 2 will be used in an other, more direct way to model the reasoning behaviour of a person who interprets rules of inference as rules-with-possible-exceptions.

§16 Plausible Validity

In Chapter 3, we assumed that the extension of a defeasible conditional is completely determined by the extensions of the antecedent and the conclusion. In fact, this extension was explicitly defined, using an implication operator on the collection of possible worlds. As said in Chapter 1, however, any attempt to define a detailed semantics of defeasible conditionals is expected to fail. The meaning of an implicational statement (that is, the user's intention) might depend on aspects of the context that were not formalized.

In this chapter, we will no longer assume that the extension of a (conditional) sentence is a subset of a collection of possible worlds. Instead, we will work with a set M, representing a person's repertoire of sentences, and we will identify a proposition like "$a \to b$" with the collection of pairs $(\alpha, \beta) \in M \times M$ for which the person considers the sentence $\alpha \to \beta$ to be true. The precise meaning of the sentences in M remains unspecified, in the same way as the exact nature and identity of possible worlds may remain unspecified in possible worlds semantics.

M is not assumed to be closed under $\wedge$, $\vee$, $\to$ and $\neg$. In order to define validity of rules like

$$\frac{a \wedge b \to c}{a \to \neg b \vee c} \qquad \text{or} \qquad \frac{a \to b}{a \wedge c \to b}$$

we will work (alongside with M) with the language L that was used in Chapter 3. To avoid confusion between formulas from L and sentences from the person's repertoire M, we will call the elements of M "possible meanings". The basic formulas from L will be treated as variables ranging over M (and M thus represents the collection of possible meanings a basic formula can have).

If M is equipped with a topology, then the ideas of Chapter 4 allow us to identify, for example, the validity of the rule

$$\frac{a \to b}{a \wedge c \to b}$$

with the statement

"for practically all $\alpha, \beta, \gamma \in M$ such that $\alpha \to \beta$ is (accepted as) true, $\alpha \wedge \gamma \to \beta$ is also (accepted as) true."

In this way, the inference rule becomes a statement about the person's behaviour. Propositions will be statements (not about worlds, but) about elements of M.

The Definitions : Models

Let $p_1, p_2, \ldots$ be an infinite row of basic formulas. For n = 1, 2, ..., let L_n denote the language generated from $p_1, \ldots, p_n$ via $\wedge, \vee, \to$ and $\neg$. Then

$$L_1 \subseteq L_2 \subseteq L_3 \subseteq \ldots$$

The union of these sets is called L.

Let M be a set.

In the sequel (in particular, in the definition below), M^n denotes the n-fold product $M \times \ldots \times M$. For every map $\sigma: \{1, \ldots, n\} \to \{1, \ldots, m\}$, $\tilde{\sigma}$ denotes the *associated substitution*, that is the unique map $L_n \to L_m$ satisfying

(for all $a, b \in L_n$)

$$\begin{aligned} \tilde{\sigma}(a \to b) &= \tilde{\sigma}(a) \to \tilde{\sigma}(b), \\ \tilde{\sigma}(a \wedge b) &= \tilde{\sigma}(a) \wedge \tilde{\sigma}(b), \\ \tilde{\sigma}(a \vee b) &= \tilde{\sigma}(a) \vee \tilde{\sigma}(b), \\ \tilde{\sigma}(\neg a) &= \neg \tilde{\sigma}(a), \end{aligned}$$

and $\tilde{\sigma}(p_i) = p_{\sigma(i)}$, for $i = 1, \ldots, n$.

For every $a \subseteq M^n$, define

$$\tilde{\tilde{\sigma}}(a) = \{ (x_1, \ldots, x_m) \in M^m \mid (x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \in a \}.$$

Then $\tilde{\tilde{\sigma}}$ denotes a map $\mathcal{P}(M^n) \to \mathcal{P}(M^m)$.

Note that if n = m and σ is a bijection, then $\tilde{\tilde{\sigma}}$ is just a permutation of coordinates.

16.1 Definition A *possible meanings assignment* (*on* M) is a row

$$(\varphi_n)_{n = 1,2,\ldots}$$

of maps $L_n \to \mathcal{P}(M^n)$ such that

i) for n, m = 1, 2, ... , and for all $a \in L_n$:

$$\varphi_{n+m}(a) = \varphi_n(a) \times M^m,$$

ii) for every injective map $\sigma : \{1, \dots , n\} \to \{1, \dots , m\}$, and for all $a \in L_n$,

$$\varphi_m(\tilde{\sigma}(a)) = \tilde{\tilde{\sigma}}(\varphi_n(a)).$$

For example, the statement $(\alpha, \beta) \in \varphi_2(p_1 \wedge p_2 \to \neg p_2)$ should be read as "α and β are two sentences such that the person considers $\alpha \wedge \beta \to \neg\beta$ to be true." Note that, in contrast with L, M is not necessarily closed under $\wedge$, $\vee$, $\to$ and $\neg$.

In view of our intentions, as presented in the beginning of this section, conditions i) and ii) above seem reasonable.
i) expresses that, for example, for every triple $(\alpha, \beta, \gamma) \in M^3$, whether the person accepts $\alpha \to \beta$ as true or not does not depend on γ.
ii) expresses that the assignment (φ_n) is *permutation-invariant* : replacing basic formulas by others does not essentially change the extension of formulas. For example, if $a = p_1 \to p_2$, $n = 2$, $m = 4$, $\sigma(1) = 4$ and $\sigma(2) = 3$, then ii) amounts to the statement

$$\varphi_4(p_4 \to p_3) = \{(x_1, \dots , x_4) \mid (x_4, x_3) \in \varphi_2(p_1 \to p_2) \}.$$

Property ii) guarantees that there is no "hidden" information in the choice of basic formulas.
Furthermore, note that i) and ii) imply, for $m \le n$,

$$\varphi_n(p_m) = M^{m-1} \times \varphi_1(p_1) \times M^{n-m}$$

(proof: use σ: $\{1\} \to \{1, \dots , n\}$, $\sigma(1) := m$ and $a := p_1$)
and that i) is a consequence of ii)
(proof: use σ: $\{1, \dots , n\} \to \{1, \dots , n+m\}$, $\sigma(k) := k$, $k = 1, \dots , n$).

16.2 Definition A *possible meanings model* is a pair $(M, (\varphi_n)_{n=1,2,\dots})$ such that M is a set and $(\varphi_n)_{n=1,2,\dots}$ is a possible meanings assignment on M.

Examples
It takes some preliminary work to give examples of possible meanings models. One possibility is to use the implication operators O(,) and U(,) that were defined in Chapter 3. The resulting models may or may not be realistic models of rational behaviour, but they are needed to prove some important results.

16.3 Definition A possible meanings model $(M, (\varphi_n)_{n = 1,2,...})$ is called *stiff* whenever, for n = 1, 2, ... , for all a, b $\in L_n$:

$$\varphi_n(a \wedge b) = \varphi_n(a) \cap \varphi_n(b),$$
$$\varphi_n(a \vee b) = \varphi_n(a) \cup \varphi_n(b),$$
$$\varphi_n(\neg a) = M^n \setminus \varphi_n(a).$$

If M is equipped with a topology, then $(M, (\varphi_n)_{n = 1,2,...})$ is called *O-stiff* whenever it is stiff and for n = 1, 2, ... , and for all a, b $\in L_n$:

$$\varphi_n(a \rightarrow b) = O(\varphi_n(a), \varphi_n(b)),$$

where O(,) is the O-operator of Chapter 3 (Definition 9.6).
Likewise, it is called *U-stiff* whenever it is stiff and for n = 1, 2, ... , and for all a, b $\in L_n$:

$$\varphi_n(a \rightarrow b) = U(\varphi_n(a), \varphi_n(b)),$$

where U(,) is the operator defined in Definition 12.1.

In the sequel we will write "let M be a possible meanings model" as shorthand for "let $(M, (\varphi_n)_{n = 1,2,...})$ be a possible meanings model and let M be equipped with some topology."

16.4 Lemma Let X and Y be topological spaces, $a, c \subseteq X$, and $b, d \subseteq Y$.
If $a \times b \rightarrow c \times d$, then either $a \rightarrow c$ or $b = \varnothing$.
Proof: If $a \times b \rightarrow c \times d$, then $a \times b \rightarrow c \times Y$ (5.10 iv),
hence $a \times b \rightarrow (a \cap c) \times b$ (5.10 ii and 5.10 iv).
By the "reverse" of the independence theorem (that is, the remark following Theorem 14.6),

$$b = \varnothing \text{ or } a \rightarrow c .$$

□

16.5 Lemma Let X and Y be topological spaces, and $a, b \subseteq X$.
Let $O_X(,)$ and $O_{X\times Y}(,)$ denote the O-operators in X and X × Y, respectively. Then

$$O_X(a, b) \times Y = O_{X\times Y}(a \times Y, b \times Y).$$

Proof:
1) $O_X(a, b) \cap a \rightarrow b$. By the independence theorem,

$$(O_X(a, b) \cap a) \times Y \rightarrow b \times Y, \text{ i.e.}$$
$$(O_X(a, b) \times Y) \cap (a \times Y) \rightarrow b \times Y.$$

Hence, $O_X(a, b) \times Y \subseteq O_{X\times Y}(a \times Y, b \times Y)$.
2) Suppose that $O_1 \times O_2 \subseteq O_{X\times Y}(a \times Y, b \times Y)$, and that $O_1 \times O_2 \neq \varnothing$.
Then $(O_1 \times O_2) \cap (a \times Y) \rightarrow b \times Y$, i.e.,

$$(O_1 \cap a) \times O_2 \rightarrow b \times Y.$$

Since $O_2 \neq \emptyset$, it follows by the reverse of the independence theorem that

$$O_1 \cap a \to b,$$

hence $O_1 \subseteq O_X(a, b)$ and $O_1 \times O_2 \subseteq O_X(a, b) \times Y$.

Hence, $O_{X\times Y}(a \times Y, b \times Y) \subseteq O_X(a, b) \times Y$.

□

We are now ready to give examples of possible meanings models.

16.6 Example If M is a topological space, and $V \subseteq M$, then there is one and only one possible meanings assignment $(\varphi_n)_{n = 1,2,\ldots}$ such that $(M, (\varphi_n)_{n=1,2,\ldots})$ is O-stiff and $\varphi_1(p_1) = V$.

Proof: Condition i) of Definition 16.1 is a straightforward consequence of Lemma 16.4. Condition 16.1 ii) is then easy to see. Note that 16.1 ii) implies that $\varphi_n(p_i) = M^{i-1} \times V \times M^{n-i}$ for n = 1, 2, ... and i = 1, ... , n. Hence, demanding stiffness guarantees that $(\varphi_n)_{n = 1,2,\ldots}$ is unique.

□

16.7 Lemma Let X and Y be topological spaces, and $a, b \subseteq X$.
Let $U_X(\ ,\ )$ and $U_{X\times Y}(\ ,\ )$ denote the U-operators in X and $X \times Y$, respectively. Then

$$U_X(a, b) \times Y = U_{X\times Y}(a \times Y, b \times Y).$$

Proof (relying on 12.2, 12.3, 14.6 and 16.4) :

If $Y = \emptyset$, then the statement is trivial. Let us therefore suppose that $Y \neq \emptyset$.

If $a \to b$ is true, then $a \times Y \to b \times Y$ is also true, and

$$U_X(a, b) \times Y = X \times Y = U_{X\times Y}(a \times Y, b \times Y).$$

If $a \to b$ is not true, then $a \times Y \to b \times Y$ is not true (since $Y \neq \emptyset$), hence

$$U_X(a, b) \times Y = (O_X(a, b) \cap \bar{a}) \times Y$$

and $U_{X\times Y}(a \times Y, b \times Y) = O_{X\times Y}(a \times Y, b \times Y) \cap \overline{a \times Y}$.

But $\overline{a \times Y} = \bar{a} \times \overline{Y} = \bar{a} \times Y$ (see B.12).

Hence, by 16.5,

$$(O_X(a, b) \cap \bar{a}) \times Y = (O_X(a, b) \times Y) \cap (\overline{a \times Y}) =$$

$$= O_{X\times Y}(a \times Y, b \times Y) \cap \overline{a \times Y}.$$

Hence, $U_X(a, b) \times Y = U_{X\times Y}(a \times Y, b \times Y)$

□

16.8 Example If M is a topological space, and $V \subseteq M$, then there is one and only one possible meanings assignment $(\varphi_n)_{n=1,2,...}$ such that (M, $(\varphi_n)_{n=1,2,...}$) is U-stiff and $\varphi_1(p_1) = V$.
(Proof: Similar to the proof of 16.6; use Lemma 16.7 instead of 16.5.)

It is not straightforward to give an example of a possible meanings model that is not stiff. Nevertheless, non-stiff models should not be regarded as uninteresting or nonsensical. For example, if the person considers $\alpha \in M$ to be a half truth, acceptable only when accompanied by $\beta \in M$, then $\alpha \wedge \beta$ is accepted, but α is not. That is :

$$\varphi_2(p_1 \wedge p_2) \ni (\alpha, \beta), \text{ but } \varphi_1(p_1) \not\ni \alpha,$$

hence

$$\varphi_2(p_1 \wedge p_2) \not\subseteq \varphi_2(p_1) \ (= \varphi_1(p_1) \times M),$$

hence, M is not stiff.
We could still imagine, though, that the person accepts

$$\frac{p_1 \wedge p_2}{p_1}$$

as-a-rule.

More Definitions : M-validity
Let M be a possible meanings model, and let $a_1, \ldots, a_m, b_1, \ldots, b_n$ be formulae from L. Say, $a_1, \ldots, a_m, b_1, \ldots, b_n \in L_N$.

16.9 Definition The rule

$$\frac{a_1, \ldots, a_m}{b_1, \ldots, b_n}$$

is called *M-valid* whenever

$$\varphi_N(a_1) \cap \ldots \cap \varphi_N(a_m) \rightarrow \varphi_N(b_1) \cap \ldots \cap \varphi_N(b_n)$$

is true in the topological space M^N.

Note that this definition does not really depend on N, since

$$\varphi_N(a_1) \cap \ldots \cap \varphi_N(a_m) \rightarrow \varphi_N(b_1) \cap \ldots \cap \varphi_N(b_n)$$

implies

$$\varphi_K(a_1) \cap \ldots \cap \varphi_K(a_m) \rightarrow \varphi_K(b_1) \cap \ldots \cap \varphi_K(b_n),$$

for all K such that $a_1, \ldots, a_m, b_1, \ldots, b_n \in L_K$.
(Proof: If $N < K$, then 16.1 i) implies

$$\varphi_K(a_i) = \varphi_N(a_i) \times M^{K-N} \text{ for } i = 1, \dots, m$$

(likewise for b_i, $i = 1, \dots, n$),

hence

$$\varphi_K(a_1) \cap \dots \cap \varphi_K(a_m) \rightarrow \varphi_K(b_1) \cap \dots \cap \varphi_K(b_n),$$

by Theorem 14.6.

If $K < N$ and $M \neq \varnothing$, use the reverse of Theorem 14.6.

If $M = \varnothing$, then $\varphi_K(a_i) = \varnothing$, for all i and all K.)

16.10 Proposition If $\sigma : \{1, \dots, N\} \rightarrow \{1, \dots, K\}$ is an injective map and $\dfrac{a_1, \dots, a_m}{b_1, \dots, b_n}$ is M-valid, then $\dfrac{\tilde{\sigma}(a_1), \dots, \tilde{\sigma}(a_m)}{\tilde{\sigma}(b_1), \dots, \tilde{\sigma}(b_n)}$ is also M-valid.

(Proof: Use 16.1 ii) and Theorem 14.6.)

16.11 Theorem Suppose that M is a possible meanings model, and that $a_1, \dots, a_m, b_1, \dots, b_n \in L_N$. Let c be a formula of L_{N+K} that does not contain any of the basic formulas $p_1, \dots, p_N$.

If the rule

$$\frac{a_1, \dots, a_m}{b_1, \dots, b_n} \text{ is M-valid, then } \frac{a_1, \dots, a_m, c}{b_1, \dots, b_n} \text{ is M-valid.}$$

Proof: If c does not contain $p_1, \dots,$ nor p_N, then

$$\varphi_{N+K}(c) = M^N \times \varphi_K(c') \text{ for some formula } c' \in L_K.$$

Note that $\varphi_{N+K}(a_i) = \varphi_N(a_i) \times M^K$ for $i = 1, \dots, m$.

Likewise for b_i, $i = 1, \dots, n$.

Hence

$$\varphi_N(a_1) \cap \dots \cap \varphi_N(a_m) \rightarrow \varphi_N(b_1) \cap \dots \cap \varphi_N(b_n)$$

implies

$$(\varphi_N(a_1) \cap \dots \cap \varphi_N(a_m)) \times \varphi_K(c') \rightarrow (\varphi_N(b_1) \cap \dots \cap \varphi_N(b_n)) \times \varphi_K(c'),$$

which implies

$$(\varphi_N(a_1) \cap \dots \cap \varphi_N(a_m)) \times \varphi_K(c') \rightarrow (\varphi_N(b_1) \cap \dots \cap \varphi_N(b_n)) \times M^K,$$

which is equivalent to

$$\varphi_{N+K}(a_1) \cap \dots \cap \varphi_{N+K}(a_m) \cap \varphi_{N+K}(c) \rightarrow (\varphi_{N+K}(b_1) \cap \dots \cap \varphi_{N+K}(b_n),$$

that is,

$$\frac{a_1, \dots, a_m, c}{b_1, \dots, b_n} \text{ is M-valid.}$$

□

16.12 Corollary If $c \in L$ has no basic formulae in common with any of the formulas $a_1, \ldots, a_m, b_1, \ldots, b_n$, and $\dfrac{a_1, \ldots, a_m}{b_1, \ldots, b_n}$ is M-valid, then

$$\frac{a_1, \ldots, a_m, c}{b_1, \ldots, b_n} \text{ is M-valid.}$$

(Proof: Use 16.10 and 16.11.)

16.13 Example If $M = \mathbb{R}$, $(M, (\varphi_n)_{n=1,2,\ldots})$ is O-stiff and $V := \varphi_1(p_1) := (0, 1) \cup (1, \infty)$, then

$$\frac{p_1 \to p_2,\ p_1,\ \neg p_2}{p_2}$$

is not M-valid.

Proof: $\varphi_2(p_1 \to p_2) = O(V \times \mathbb{R}, \mathbb{R} \times V) = \{ (x, y) \in \mathbb{R}^2 \mid x < 0 \text{ or } y > 0 \}$.

But $\{ (x, y) \in \mathbb{R}^2 \mid (x < 0 \text{ or } y > 0) \text{ and } x \in V \text{ and } y \notin V \} =$

$\{ (x, y) \in \mathbb{R}^2 \mid y = 1 \text{ and } x \in V \} = V \times \{1\}$,

and $V \times \{1\} \not\twoheadrightarrow V \times V$. Hence, $V \times \{1\} \not\twoheadrightarrow M \times V$.

Hence, $\varphi_2(p_1 \to p_2) \cap \varphi_2(p_1) \cap \varphi_2(\neg p_2) \not\twoheadrightarrow \varphi_2(p_2)$.

□

Note that $\dfrac{p_1 \to p_2,\ p_1,\ \neg p_2}{\neg p_2}$ is M-valid, as well as $\dfrac{p_1 \to p_2,\ p_1}{p_2}$ (see also Theorem 16.14 below).

16.14 Theorem If M is an O-stiff possible meanings model, then every rule that is O-valid (see Definition 9.11) is M-valid.

Proof:

Suppose that $(M, (\varphi_n)_{n=1,2,\ldots})$ is O-stiff, that $a_1, \ldots, a_m, b_1, \ldots, b_n \in L_N$, and that

$$\frac{a_1, \ldots, a_m}{b_1, \ldots, b_n}$$

is O-valid.

Let (X, O, φ) be an extended model such that $X = M^N$ and $\varphi(p_i) = \varphi_N(p_i)$ for $i = 1, \ldots, N$. Since M is O-stiff,

$\varphi_N(a_i) = \varphi(a_i)$, for $i = 1, \ldots, m$.

Likewise for b_i, $i = 1, \ldots, n$.

Since $\dfrac{a_1, \ldots, a_m}{b_1, \ldots, b_n}$ is O-valid,

$$\varphi(a_1) \cap ... \cap \varphi(a_m) \;\rightarrow\; \varphi(b_1) \cap ... \cap \varphi(b_n).$$

Hence,

$$\varphi_N(a_1) \cap ... \cap \varphi_N(a_m) \;\rightarrow\; \varphi_N(b_1) \cap ... \cap \varphi_N(b_n),$$

that is,

$\dfrac{a_1, \; \dots \; , a_m}{b_1, \; \dots \; , b_n}$ is M-valid.

$\square$

The reverse of this theorem is not true :

16.15 Corollary If M is an O-stiff possible meanings model, and a, b, c are three distinct basic formulas from L, then

$$\frac{a \rightarrow b, \quad a, \quad c}{b}$$

is M-valid, but not O-valid.

Proof: By 16.14 and 9.12, $\dfrac{a \rightarrow b, \quad a}{b}$ is M-valid. The corollary now follows from Theorem 16.11 and Example 11.11 ii).

$\square$

A Closer Look at U-stiff Models

16.16 Theorem If M is a U-stiff possible meanings model, then every rule that is U-valid (see Definition 12.4) is M-valid.

Proof: Similar to the proof of 16.14.

$\square$

Note that $\dfrac{p_1 \rightarrow p_2, \quad p_1}{p_2}$ is O-valid as well as U-valid (see Proposition 12.8).

$\dfrac{p_1 \rightarrow p_2}{p_1 \wedge p_3 \rightarrow p_2}$ is O-valid, but not U-valid (see Proposition 12.5).

16.17 Proposition Let M be a U-stiff possible meanings model, and V := $\varphi_1(p_1)$. Then the following two statements are equivalent :

i) $\dfrac{p_1 \rightarrow p_2}{p_1 \wedge p_3 \rightarrow p_2}$ is M-valid,

ii) V is dense in M or V^c is dense in M.

Proof: If V is full in M, then (by Theorem 14.6 and 5.10 iv)

$$V \times M \to M \times V \text{ and } V \times M \times V \to M \times V \times M,$$

hence $\varphi_3(p_1 \to p_2) = M^3 = \varphi_3((p_1 \wedge p_3) \to p_2)$.
Likewise, if $V = \varnothing$, then $\varphi_3(p_1 \to p_2) = M^3 = \varphi_3((p_1 \wedge p_3) \to p_2)$.
Hence, we may assume without loss of generality that $M \not\to V$ and $V \neq \varnothing$.

If $M \not\to V$ and $V \neq \varnothing$, then (by 16.4)

$$V \times M \not\to M \times V \text{ and } V \times M \times V \not\to M \times V \times M,$$

hence (using 12.2 and 12.3)

$$\varphi_3(p_1 \to p_2) = O(V \times M, M \times V) \times M \cap \overline{V} \times M \times M.$$

and $\varphi_3((p_1 \wedge p_3) \to p_2) = O(V \times M \times V, M \times V \times M) \cap \overline{V} \times M \times \overline{V}$.
(Note that $\varphi_3(p_1 \wedge p_3) = V \times M \times V$ and that $\overline{V \times M \times V} = \overline{V} \times M \times \overline{V}$ by B.12.)

But $O(V \times M, M \times V) \times M \subseteq O(V \times M \times V, M \times V \times M)$.
(To see this : for all $O \subseteq M \times M$,

$$O \cap (V \times M) \to M \times V \text{ implies (by 14.6 and 5.10 iv)}$$
$$(O \times M) \cap (V \times M \times V) \to M \times V \times M.\ )$$

Hence, $\varphi_3(p_1 \to p_2) \subseteq O(V \times M \times V, M \times V \times M)$.
Hence,

$$\varphi_3(p_1 \to p_2) \to \varphi_3((p_1 \wedge p_3) \to p_2)$$

is true (only) if

$$O(V \times M, M \times V) \times M \cap (\overline{V} \times M \times M) \to \overline{V} \times M \times \overline{V},$$

i.e. $(O(V \times M, M \times V) \cap (\overline{V} \times M)) \times M \to \overline{V} \times M \times \overline{V}$.

It easily follows from 14.6 and 16.4, that this is true (only) if
either $O(V \times M, M \times V) \cap (\overline{V} \times M) = \varnothing$,
or $M \to \overline{V}$.
Of these two statements, the latter is true (only) if V is dense in M.
To see that the the former is true (only) if V^c is dense in M, we first note that $O(V \times M, M \times V)$ is an open set and that $\overline{V} \times M = \overline{V \times M}$ (see B.12).
Hence, by the definition of closure,

$$O(V \times M, M \times V) \cap (\overline{V} \times M) = \varnothing$$

(only) if

$$O(V \times M, M \times V) \cap (V \times M) = \varnothing.$$

Now suppose that V^c is dense in M.
Then for all $O_1, O_2 \subseteq M$,

$$O_1 \times O_2 \cap (V \times M) \to M \times V, \text{ i.e.,}$$
$$(O_1 \cap V) \times O_2 \to M \times V,$$

is equivalent to

$$(O_1 \cap V) \times O_2 \to (O_1 \cap V) \times V,$$

By 14.6 and 16.4, it is easy to see that the latter is true (only) if

$$O_1 \cap V = \varnothing \quad \text{or} \quad O_2 \to V,$$

which is equivalent to $O_1 \cap V = \varnothing$, since V^c is dense in O_2.

Which proves that $O(V \times M, M \times V) \subseteq V^c \times M$, i.e.,

$$O(V \times M, M \times V) \cap (V \times M) = \varnothing.$$

On the other hand, if V^c is not dense in M,
say $O \neq \varnothing$ and $O \cap V^c = \varnothing$, then

$$(O \times O) \cap (V \times M) \to M \times V,$$

hence,

$$O \times O \subseteq O(V \times M, M \times V)$$

and $(O \times O) \cap (V \times M) \neq \varnothing$.

Hence, $O(V \times M, M \times V) \cap (V \times M) \neq \varnothing$.

□

16.18 Example Let M be a U-stiff possible meanings model, such that $V := \varphi_1(p_1)$ is full in M, $V \neq \varnothing$ and $V \neq M$. (For example, $M = \mathrm{IR}$ and $V = \mathrm{IR} \setminus \{0\}$.)

Then $\dfrac{p_1 \to p_2}{p_1 \wedge \neg p_2 \to p_2}$ is not M-valid.

Proof: Since V is full in M, $V \times M \to M \times V$ is true (14.6 and 5.10 iv), hence

$$\varphi_3(p_1 \to p_2) = U(V \times M, M \times V) = M^2.$$

Since $V^c \to V$ is not true, and $V \neq \varnothing$,

$V \times V^c \to M \times V$ is not true (by 16.4),

hence, $\varphi_3(\,(p_1 \wedge \neg p_2) \to p_2) = U(V \times V^c, M \times V) =$

$$= O(V \times V^c, M \times V) \cap \left(\overline{V \times V^c}\right).$$

Hence, $\varphi_3(\,(p_1 \wedge \neg p_2) \to p_2) \subseteq \left(\overline{V \times V^c}\right)$.

Hence,

$\varphi_3(p_1 \to p_2) \to \varphi_3(\,(p_1 \wedge \neg p_2) \to p_2)$

would imply $\varphi_3(p_1 \to p_2) \to \left(\overline{V \times V^c}\right)$,

i.e., $M^2 \to \left(\overline{V \times V^c}\right) = \overline{V} \times \overline{V^c}$ (see B.12),

which would imply (by 14.6 and 16.4)

$M \to \overline{V^c}$, hence $M = \overline{V^c}$, i.e., V^c is dense in M.

But V is full in M, and $M \neq \varnothing$.
Hence,

$$\varphi_3(p_1 \to p_2) \to \varphi_3((p_1 \wedge \neg p_2) \to p_2)$$

is not true.

□

Plausible Validity

Example 16.18 shows that the statement

$$\text{"}\frac{p_1 \to p_2}{p_1 \wedge p_3 \to p_2} \quad \text{is M-valid"}$$

is strictly weaker than the statement

$$\text{"for all a, b, c} \in L: \frac{a \to b}{a \wedge c \to b} \quad \text{is M-valid."}$$

However, if a person's repertoire M satisfies the former, but not the latter, it is still fair to say that the person obeys-up-to-possible-exceptions the rule of monotony, since it amounts to the statement:
"for practically all $\alpha, \beta, \gamma \in M$ such that $\alpha \to \beta$ is (accepted as) true, $\alpha \wedge \gamma \to \beta$ is also (accepted as) true."
Likewise, the M-validity of, say,

$$\frac{p_1 \to p_2,\ p_1 \to p_3}{p_1 \to p_2 \wedge p_3}$$

is enough to say that the person obeys (up-to-possible-exceptions) the "and-introduction rule". In view of the remarks of §10, this should make us curious *which laws are obeyed by all persons obeying the rules of classical propositional logic*. Hence, let us define :

16.19 Definition A rule is called *plausibly valid* whenever it is M-valid in all possible meanings models M in which the following rules are M-valid : *(where a, b, c and d denote four distinct basic formulas)*

$$\frac{c \to a,\ c \to b}{c \to a \wedge b} \qquad \frac{c \to a \wedge b}{c \to a} \quad \frac{c \to a \wedge b}{c \to b}$$

$$\frac{c \to a}{c \to a \vee b} \quad \frac{c \to b}{c \to a \vee b} \qquad \frac{c \to a \vee b,\ c \wedge a \to d,\ c \wedge b \to d}{c \to d}$$

$$\frac{c \wedge a \to b}{c \to \neg a \vee b} \qquad \frac{c \to a,\ c \to \neg a \vee b}{c \to b}$$

$$\frac{c \wedge a \to b,\ c \wedge a \to \neg b}{c \to \neg a} \qquad \frac{c \to a,\ c \to \neg a}{c \to b}$$

$$\frac{}{a \to a} \qquad \frac{c \wedge a \wedge b \to d}{c \wedge b \wedge a \to d} \qquad \frac{c \wedge a \wedge a \to b}{c \wedge a \to b}$$

$$\frac{a \to b,\ a}{b} \qquad \frac{c \to a}{c \wedge b \to a}$$

(cf. Theorem 9.15)

16.20 Corollary If a, b, c are three distinct basic formulas, then

$\dfrac{a \to b,\quad a,\quad c}{b}$ is plausibly valid, $\dfrac{a \to b,\quad a,\quad \neg b}{b}$ is not.

$\dfrac{a \to b}{a \wedge c \to b}$ is plausibly valid, $\dfrac{a \to b}{a \wedge \neg b \to b}$ is not.

Proof: Direct corollaries of previous results (16.11 (+16.19), 16.12, 16.19, and 16.18, respectively).

□

16.21 Corollary If $c \in L$ has no basic formulae in common with any of the formulas $a_1, \ldots, a_m, b_1, \ldots, b_n$, and $\dfrac{a_1, \ldots, a_m}{b_1, \ldots, b_n}$ is plausibly valid, then $\dfrac{a_1, \ldots, a_m, c}{b_1, \ldots, b_n}$ is plausibly valid.

Proof: Direct corollary of Theorem 16.11.

□

16.22 Corollary Suppose that $a_1, \ldots, a_m, b_1, \ldots, b_n \in L_N$ and that $\sigma : \{1, \ldots, N\} \to \{1, \ldots, K\}$ is an injective map.

If $\dfrac{a_1, \ldots, a_m}{b_1, \ldots, b_n}$ is plausibly valid, then $\dfrac{\tilde{\sigma}(a_1), \ldots, \tilde{\sigma}(a_m)}{\tilde{\sigma}(b_1), \ldots, \tilde{\sigma}(b_n)}$ is also plausibly valid.

Proof: Direct corollary of Proposition 16.10.

□

Bold Validity

16.23 Definition A rule is called *boldly valid* whenever it is M-valid in all *stiff* possible meanings models M in which the rules of Definition 16.19 are M-valid.

Then results similar to 16.20, 16.21 and 16.22 can still be proved, as well as the following :

16.24 Proposition If $a_0, a_1, \dots , a_n$ do not contain the symbol "$\rightarrow$", then the following two statements are equivalent :

i) $\dfrac{a_1, \dots , a_n}{a_0}$ is boldly valid,

ii) $a_1, \dots , a_n \vdash a_0$ holds classically.

Proof: Suppose that $a_0, a_1, \dots , a_n \in L_N$ do not contain "$\rightarrow$".

1) If $a_1, \dots , a_n \vdash a_0$, then in every stiff model M,

$$\varphi_N(a_1) \cap \dots \cap \varphi_N(a_n) \subseteq \varphi_N(a_0),$$

hence,

$$\varphi_N(a_1) \cap \dots \cap \varphi_N(a_n) \rightarrow \varphi_N(a_0),$$

hence,

$\dfrac{a_1, \dots , a_n}{a_0}$ is M-valid.

2) If $\dfrac{a_1, \dots , a_n}{a_0}$ is boldly valid, then it is also M-valid in the (unique) O-stiff model with $M = \{0,1\}$ (discrete topology) and $V = \varphi_1(p_1) = \{1\}$.

And it is easy to see that M-validity of $\dfrac{a_1, \dots , a_n}{a_0}$ in that model is equivalent to $a_1, \dots , a_n \vdash a_0$.

□

§17 Evaluation

With the definition of bold validity, this thesis has come to an end. It is to be noted, though, that it is not our intention to propose any of the formalisms in this thesis as a concrete description (or prescription) of an inference engine, as is usually the goal in other approaches towards non-monotonic logic. They are intended to be an aid in understanding the nature of non-monotony in practical reasoning (and not, in the first place, to invent better systems of artificial reasoning).

The notion of bold validity is a formal description of an inference relation on classical propositional logic extended with a binary connective "$\rightarrow$" that represents defeasible implication. (That it is an extension of the classical inference relation follows from Proposition 16.24.) Bold validity, interpreted as a description of a person displaying non-monotonic reasoning behaviour, is more convincing than topological validity (as defined in Chapter 3).

In the first place, the core of the definition (namely, the definition of M-validity), is a more direct representation of validity-as-a-rule-with-possible-exceptions. The M-validity of a rule amounts to a (non-standard) universally quantified statement about sentences, instead of worlds. For example, the validity of the rule

$$\frac{a \rightarrow b,\ a}{b}$$

is interpreted as "for practically all sentences α and β from the person's repertoire such that the person accepts $\alpha \rightarrow \beta$ and α as true, β is accepted as well."

In the second place, no fixed semantics of implication was assumed, in that the exact nature of the elements of M was left unspecified, as well as the exact content of statements like

$$(\alpha, \beta) \in \varphi_2(p_1 \rightarrow p_2).$$

The concrete semantical idea of Chapter 2 (Definition 5.7) was used *only* to define what it means to accept a rule of inference as a defeasible rule. Although our only concrete examples of possible meanings models used explicit definitions, like O(,) or U(,), other, more flexible, models were not excluded (see the definitions of plausible and bold validity, Definitions 16.19 and 16.23).

In fact, since we did not assume M to be closed under $\wedge$, $\vee$, $\rightarrow$ and $\neg$, we did not exclude persons with a "flexible" interpretation of implication. For example, in the formula

$$p_3 \wedge (p_1 \rightarrow p_2),$$

the person might use different interpretations of implication, depending on the concrete interpretation of p_3. This represents the possibility that, for two concrete sentences α and β, the meaning of $\alpha \to \beta$ might depend on more than just α and β. On the other hand, by demanding stiffness, we did restrict the possible flexibility of the connectives $\wedge$, $\vee$ and $\neg$. The notion of plausible validity does not even have this restriction.
In the third place,

$$\frac{a \to b}{a \wedge c \to b} \quad \text{is boldly valid, but} \quad \frac{a \to b}{a \wedge \neg b \to b} \quad \text{is not.}$$

Likewise,

$$\frac{a \to b, \quad a, \quad c}{b} \quad \text{is boldly valid, but} \quad \frac{a \to b, \quad a, \quad \neg b}{b} \quad \text{is not.}$$

Hence, a person whose inference relation matches with bold validity displays more natural reasoning behaviour than a person that reasons according to topological validity (see Example 11.11). For this reason, bold validity might be easier acknowledged as a formal description of a person "accepting the rules of classical propositional logic as rules-with-possible-exceptions."
Nevertheless, the approach of §16 resembles the approach of Chapter 3, in that the notion of full subset is still the core of the system, and that the system arises in a natural way from this notion.

Comparison with other approaches
Although, as said, neither plausible validity nor bold validity is proposed as a "definitive" system, it is interesting to compare the behaviour of these notions with some of the existing approaches. Since our system does not address matters of causality or inertia in changing domains, we will restrict our attention to a number of systems that propose solutions to the irrelevance problem or related issues.

Preferential entailment (see [LM 92]) is an approach in which, given a set Δ of defaults, a new set Δ' of defaults is defined, called the *rational closure* of Δ. Preferential entailment solves the irrelevance problem, in that the rational closure of $\Delta = \{a \mid\!\sim b\}$ contains $a \wedge c \mid\!\sim b$ if c is logically independent from a and b, but it does not contain $a \wedge \neg b \mid\!\sim b$. That is, in the presence of the rule $a \mid\!\sim b$ and assumptions a and c, preferential entailment allows the conclusion b, whenever c is independent from a and b, but not if $c \equiv \neg b$.

However, if b, c, f, p, and w are five distinct basic formulas, then in the presence of the rules $b \mathrel{|\!\sim} f$ (birds fly), $p \mathrel{|\!\sim} b \wedge \neg f$ (penguins are birds that do not fly), $c \mathrel{|\!\sim} w$ (crows have wings) and the assumptions p and c, preferential entailment fails to conclude w, as [Asher 95] correctly points out. Using 16.21, on the other hand, it is easy to see that

$$\frac{b \to f, \quad p \to b \wedge \neg f, \quad c \to w, \quad p, \quad c}{w} \tag{1}$$

is plausibly valid as well as boldly valid (since $\dfrac{c \to w, \quad c}{w}$ is plausibly valid and none of the other premisses contains c or w.)
Likewise, in the presence of the rules $a \mathrel{|\!\sim} b$, $c \mathrel{|\!\sim} d$ and the assumptions a, $\neg b$ and c, preferential entailment does not conclude d, but

$$\frac{a \to b, \quad c \to d, \quad a, \quad \neg b, \quad c}{d} \tag{2}$$

is easily seen to be plausibly valid as well as boldly valid.
In [Benferhat *et al.* 93], these and similar deficiencies of preferential entailment were called "the drowning effect", and a rather ad hoc solution was presented. A more fundamental solution (from the viewpoint of mathematical logic) was given in [GMP 93], using probabilistic notions (in particular, entropy).
[Asher 95] also mentions

$$\frac{a \to b, \quad a \to c, \quad a, \quad \neg b}{c} \tag{3}$$

and

$$\frac{b \to f, \quad p \to b \wedge \neg f, \quad b \to e, \quad p}{e}. \tag{4}$$

It is not known whether one or both of these rules is plausibly valid or boldly valid. Preferential entailment does not sanction these inferences.
We will return to (3) and (4) in the next subsection.

Each of these three approaches (i.e., [LM 92], [Benferhat et al 93], and [GMP 93]) is restricted to inferences of this simple kind. Unlike bold validity, they cannot be seen as descriptions of "classical propositional logic extended with a defeasible conditional," since the defeasible conditional is not a (nestable) connective in these formalisms.

On the other hand, *commonsense entailment* (see [Asher & Morreau 91] or [Asher 95]), an attempt to formalize "the calculus of generics", does define a non-monotonic entailment relation on a language having a defeasible conditional as a connective. The approach solves (1) and (2); it solves (3) and (4) only after considerable changes (see [Asher 95]). In general, however, their system is not "cumulative", in contrast with ours :

If $\dfrac{a_1, \ldots, a_n}{b}$ and $\dfrac{a_1, \ldots, a_n, b}{c}$ are plausibly valid (boldly valid), then $\dfrac{a_1, \ldots, a_n}{c}$ is also plausibly valid (boldly valid).

(Proof: Direct corollary of the Definitions 16.19 (16.23), 16.9 and Proposition 5.11 ii.)

Although this might seem to be a point in our favour, it is questionable whether the inference relation should be cumulative. After all (see [GMP 93]),

$$\frac{a \to b, \quad b \to c}{a \to c} \quad \text{and} \quad \frac{a \to b, \quad b \to c, \quad a \to c}{a \wedge \neg b \to c} \quad \text{should be valid,}$$

but $\dfrac{a \to b, \quad b \to c}{a \wedge \neg b \to c}$ should not be valid.

Although there is little difference between "the calculus of generics" and "the calculus of rules-with-possible-exceptions", the approach of §16 is very different from commonsense entailment. We assume that the axioms or principles governing the calculus are themselves rules-with-possible-exceptions. Hence, a formal characterization of this calculus requires not only a set of axioms, but also (and in the first place) an adequate formalism that handles these axioms as rules-with-possible-exceptions. We could say that the latter is completed with the introduction of the notions of §16. However, the choice of a suitable set of axioms leaves room for discussion.

Possible Further Research

In §16, we simply used the ordinary axioms of classical propositional logic as the "axiom set". As we saw, that choice gave a number of satisfying results. However, it is certainly not the only possibility. Let us consider an example.

We interpreted the rule of monotony as

"for practically all $\alpha, \beta, \gamma \in M$ such that $\alpha \to \beta$ is (accepted as) true, $\alpha \wedge \gamma \to \beta$ is also (accepted as) true."

However, according to Chapter 4, this is not equivalent to

"for practically all $\alpha, \beta \in M$ such that $\alpha \to \beta$ is true :
for practically all $\gamma \in M$: $\alpha \wedge \gamma \to \beta$ is also true."

This reformulation seems acceptable as an alternative for the former interpretation of monotony. We could even consider

"for *all* $\alpha, \beta \in M$ (without exceptions) such that $\alpha \to \beta$ is true :
for practically all $\gamma \in M$: $\alpha \wedge \gamma \to \beta$ is also true."

Moreover, one could add one or more axioms, for example,

$$\frac{a \to \neg b}{\neg(a \to b)}$$

Or even seemingly bizar rules, like

$$\frac{a \wedge b, \quad \neg a}{a \to \neg b}$$

(To defend this rule, we could audaciously reason as follows : if α and β are sentences such that $\alpha \wedge \beta$ is acceptable, but α is not, then "apparently", α is a half truth, acceptable only when accompanied by β. The only situation in which it makes sense to accept $\alpha \wedge \beta$ as well as $\neg\alpha$, is that α suggests something that is known to be not true, because β is also true. That is, there exists a γ such that $\alpha \to \gamma$ and $\alpha \wedge \beta \to \neg\gamma$ are true. Audaciously accepting R6 (an ordinary consequence of P1-P5, see §8), this implies $\alpha \to \neg\beta$.)

Coming back to the rules (3) and (4), these inferences seem to be applications of the "general principle" of Chapter 4 (see p. 92). Slightly reformulating (4), for example, the underlying argument seems to be the following.

The inference of $p \to e$ (penguins have eyes) from $b \to f$ (birds fly), $p \to b$, $p \not\to f$ and $b \to e$ is correct, since $p \to b$ suggests the existence of a rule

"for practically all Z such that $b \to Z$: $p \to Z$",

$Z = f$ constitutes an exception to this rule, and $Z = e$ is an independent new instance. Hence, by the general principle of Chapter 4, we can apply the rule to $Z = e$: $b \to e$ yields $p \to e$.

Likewise, the idea behind accepting (3) seems to be the following.

The inference of c from $a \to b$, $a \to c$, a and $\neg b$ is correct, since "a" suggests the existence of a rule

"for practically all Z such that $a \to Z$: Z",

$Z = b$ constitutes an exceptions to this rule, and $Z = c$ is an independent, new instance. Hence, by the general principle of Chapter 4, we may apply the rule to $Z = c$: $a \to c$ yields c.

A very interesting task for the future would be to find "direct" translations (in the spirit of §16) of these reasoning patterns.

Other important issues to deal with are of a more fundamental nature. For example, reconsidering the proof of Proposition 16.17, we can prove that in every U-stiff model M in which the rule of monotony is M-valid, the rule of monotony is, in fact, "strictly" M-valid, in that

$$\varphi_3(p_1 \to p_2) \subseteq \varphi_3((p_1 \wedge p_3) \to p_2).$$

This means that the person accepts $\alpha \wedge \gamma \to \beta$ for *all* $\alpha, \beta, \gamma \in M$ (*without exceptions*) for which $\alpha \to \beta$ is accepted. Nevertheless, in some of these models, the rule

$$\frac{p_1 \to p_2}{p_1 \wedge \neg p_2 \to p_2}$$

is not M-valid (see Example 16.18).

We could say that in such M, monotony is valid-without-exceptions (that is, exceptions in M), but does mysteriously have exceptions "in L".

In this context, it is well to remember that the conditions 16.1 i) and ii) are rather weak. We might wonder whether it is possible to add one or more plausible conditions to Definition 16.1, in order to prevent oddities of this kind. (Demanding M to be closed under $\wedge$, $\vee$, $\neg$, and $\to$ would be an unreasonably strong condition, since this would exclude the possibility of a "flexible" meaning of implication, as explained earlier in this section.)

Recapitulation

The definitions of §16 enable us to study non-monotonic logic as an ordinary and respectable discipline of mathematical logic.

Summary and Conclusions

Although non-monotonic logic arose as a subdiscipline of artificial intelligence, the potential of the subject for the field of mathematical logic should not be underestimated, as non-monotony is connected with the most important difficulties in formalizing practical reasoning. Practical reasoning seems to be governed by general principles that function as rules-with-possible-exceptions. Hence, mathematical axioms, when interpreted in the usual way, will not suffice to describe the underlying calculus.
In this thesis, the expressive power of the language used to formulate principles of reasoning has been extended to enable the interpretation of formal axioms as valid-up-to-possible-exceptions in a number of ways. The core of each system is the notion of a full subset (Definition 5.3), a topological generalization of intuitions stemming from Euclidean geometry (a subject that preceded Aristotelean logic). This notion gives rise to a topological semantics of defeasible implication in a natural way (Definitions 5.7 and 6.1). A sound and complete axiomatization is established, relying for the proof on a result of [KLM 90] (in fact, we find the same axiom system).

The notion of a full subset is then used to define semantics of nested implicational statements (Chapter 3). This semantics is associated with two issues. In the first place, we think of an imaginary person that handles (some) rules of inference as rules-with-possible-exceptions (see §10). With the semantics defined in §9, the person turns out to obey-up-to-possible-exceptions every law of classical propositional logic, including modus ponens and the rule of monotony. Thus, it is shown that it is possible for a non-monotonic formalism to obey-up-to-possible-exceptions the rule of monotony.
A much more interesting conclusion is that the rules of classical propositional logic do apparently not determine the usual interpretation of implication (and inference) as implication-without-exceptions.
In the second place, the formalism can be seen as extending classical propositional logic with a binary connective denoting "defeasible implication." Seen in this way, the non-monotony of the inference relation has nothing to do with "non-deductive reasoning" or "jumping to conclusions", but is a necessary consequence of the presence of a defeasible conditional in the language; correct, deductive reasoning from defeasible premisses *is* non-monotonic. (More provocatively: correct, deductive reasoning in general is non-monotonic.)

The setup of Chapter 3, using nested implicational statements, is not entirely convincing, however. It does not constitute a direct interpretation of

defeasible rules-of-inference as (say) quantificational statements about sentences. (This is important, since the conclusions above might be consequences of technical peculiarities, cf. §12: U-validity, O_π-validity and S-validity do not support those conclusions.) Moreover, the behaviour of the formalism is not in all respects in accordance with typical human reasoning behaviour (see Example 11.11). The construction of formalisms that behave better in this respect requires a more general setup.
Fortunately, the notion of a full subset can also be used to interpret (universally) quantificational statements up-to-possible-exceptions, see Chapter 4. This idea provides some useful insights for practical reasoning in itself (see §15) and can indeed be used to give a direct interpretation of defeasible rules of inference (see §16). Two easily achieved inference relations (bold validity and plausible validity) reinforce the conclusions of Chapter 3, by showing that the rules of classical propositional logic do not lead to "ordinary" propositional logic if these rules are interpreted as rules-with-possible-exceptions. That is, the axiom system that is usually thought to characterize, a.o., the classical notions of inference and deducibility, presupposes "the right understanding" of these notions.
Moreover, the approach of §16 is more general than the one of Chapter 3, in that it is not limited to the collection of rules of classical propositional logic. The choice and investigation of other "sets of rules", however, is beyond the scope of this thesis.

Readers who want to know more without reading the whole thesis are advised to read the first sections of each chapter, viz. §1, §5, §9 (+ §10), §14 and §16.

Appendix A
Classical Propositional Logic

This appendix contains standard definitions and results from elementary Boolean logic, in particular, concerning Boolean algebras. Some of the proofs have been (partly) skipped. The reader is referred to any standard textbook on Boolean algebras, for example [Abbott 69].

A.1 Definition Let X be a set. A *partial ordering* (*on* X) is a binary relation $\leq$ satisfying:

i) for all $p \in X$: $p \leq p$,

ii) if $p, q, r \in X$, $p \leq q$ and $q \leq r$, then $p \leq r$,

iii) if $p, q \in X$, $p \leq q$ and $q \leq p$, then $p = q$.

A.2 Definition A *Boolean algebra* is a structure $(B, \leq, \wedge, \vee, \neg, \bot, \top)$, where B is a set, $\leq$ is a partial ordering on B, $\wedge$ and $\vee$ are binary operations on B, $\neg$ is a unary operation on B, and $\bot$ and $\top$ are elements of B, such that

i) for all $a, b, c \in B$: $a \leq b$ and $a \leq c$ (only) if $a \leq b \wedge c$,

ii) for all a, a', b, c ($\in B$) :

$$b \wedge (a \vee a') \leq c \text{ (only) if } b \wedge a \leq c \text{ and } b \wedge a' \leq c,$$

iii) for all a, b, c : if $b \wedge a \leq c$ and $b \wedge (\neg a) \leq c$, then $b \leq c$,

iv) for all a, b, c : if $a \leq b$ and $a \leq \neg b$, then $a \leq c$,

v) for all a : $\bot \leq a$ and $a \leq \top$.

A.3 Example If X is a set, then $(\mathcal{P}(X), \subseteq, \cap, \cup, .^c, \emptyset, X)$ is a Boolean algebra.

Instead of "the Boolean algebra $(B, \leq, \wedge, \vee, \neg, \bot, \top)$", we will typically write "the Boolean algebra $(B, \leq)$" or "the Boolean algebra B", if there is no danger of confusion.

A.4 Definition Let B_1 and B_2 be Boolean algebras (we will write $\leq_1$, $\leq_2$, $\wedge_1$, $\wedge_2$, etc.). A *Boolean translation* from B_1 to B_2 is a map $\varphi: B_1 \to B_2$ satisfying :

i) for all $a, b \in B_1$ such that $a \leq_1 b$: $\varphi(a) \leq_2 \varphi(b)$,

ii) for all $a, b \in B_1$: $\varphi(a \wedge_1 b) = \varphi(a) \wedge_2 \varphi(b)$,

iii) for all $a, b \in B_1$: $\varphi(a \vee_1 b) = \varphi(a) \vee_2 \varphi(b)$,

iv) for all $a \in B_1$: $\varphi(\neg_1(a)) = \neg_2(\varphi(a))$.

A.5 Typical Associations and Notations

If B is a Boolean algebra, X is a set and φ: $B \to \mathcal{P}(X)$ is a Boolean translation, then the elements of B are sometimes thought of as sentences or *propositions*. The elements of X are typically called *possible worlds*.

If $a \in B$ and $w \in X$, we say that

$w \models_{(X,\varphi)} a$ ("a is *true in* world w")

whenever $w \in \varphi(a)$.

This is sometimes abbreviated to $w \models_X a$, $w \models_\varphi a$ or even to $w \models a$, if there is no danger of confusion.

Hence, for all $a \in B$,

$\varphi(a) = \{ w \in X \mid w \models a \}$,

"the set of all possible worlds in which a is true. For this reason, $\varphi(a)$ is sometimes called the *extension* of the proposition a.

Furthermore, if a and b are elements of B, we say

$a \models_{(X,\varphi)} b$

whenever

$\varphi(a) \subseteq \varphi(b)$.

This is sometimes abbreviated to $a \models_X b$, $a \models_\varphi b$ or $a \models b$.

Note that with these notations, the properties A.4 i), ii), iii) and iv) amount to:

i) for all $a, b \in B$: if $a \le b$, then $a \models_X b$,

ii) for all $a, b \in B$, $w \in X$: $w \models a \wedge b$ (only) if $w \models a$ and $w \models b$,

iii) for all $a, b \in B$, $w \in X$: $w \models a \vee b$ (only) if $w \models a$ or $w \models b$,

iv) for all $a \in B$, $w \in X$: $w \models \neg a$ (only) if $w \not\models a$,

respectively.

A.6 Definition If φ: $B_1 \to B_2$ is a bijective Boolean translation, then it is called a *(Boolean) isomorphism*. If there exists a Boolean isomorphism φ: $B_1 \to B_2$, then B_1 is called *isomorphic with* B_2.

A.7 Proposition If B is a *finite* Boolean algebra (that is, if the underlying set B is finite), then there is a (finite) set X such that B is isomorphic with $\mathcal{P}(X)$.

Proof: Define $X = \{ p \in B \mid \text{for all } q \in B, \text{ if } q \le p, \text{ then } q = \bot \text{ or } q = p \}$.

(The elements of this set are called the *atoms* of B.)

Define for all $a \in B$:

$\varphi(a) = \{ p \in X \mid p \le a \}$.

It is elementary (but tedious) to check that φ: B $\rightarrow \mathcal{P}(X)$ is a Boolean translation, and that it is a bijection. Hence, B is isomorphic with $\mathcal{P}(X)$. Moreover, since B is finite and $X \subseteq B$, X is finite.

□

In general :

A.8 Proposition If B is a Boolean algebra, then there is a set X and a Boolean translation $\varphi: B \rightarrow \mathcal{P}(X)$ such that φ is an injective map.

Proof: Let S denote the Boolean algebra $\mathcal{P}(\{0\})$. Then $S = \{\perp, \top\}$.

Let X be the set of all Boolean translations $B \rightarrow S$.

Define for all $a \in B$:

$$\varphi(a) = \{ \tau \in X \mid \tau(a) = \top \}.$$

Then the map $\varphi: B \rightarrow \mathcal{P}(X)$ is a Boolean translation, as is easy to check.

The proof that φ is injective requires the axiom of choice, and is therefore skipped.

□

A.9 Definition / Construction / Example Let V be any set.

Let L be the language generated via $\wedge$, $\vee$, and $\neg$, with the elements of V as basic formulas, and two extra basic formulas, $\perp_L$ and $\top_L$.

Then for every Boolean algebra B and for every map f: $V \rightarrow B$, there is one and only one map $\tilde{f}: L \rightarrow B$ satisfying

i) for all $v \in V$: $\tilde{f}(v) = f(v)$,

ii) for all $a, b \in L$: $\tilde{f}(a \wedge b) = \tilde{f}(a) \wedge_B \tilde{f}(b)$,

iii) for all $a, b \in L$: $\tilde{f}(a \vee b) = \tilde{f}(a) \vee_B \tilde{f}(b)$,

iv) for all $a \in L$: $\tilde{f}(\neg a) = \neg_B(\tilde{f}(a))$,

v) $\tilde{f}(\perp_L) = \perp_B$ and $\tilde{f}(\top_L) = \top_B$.

For $a, b \in L$, define

$$a \vdash b \;:\Leftrightarrow \text{for every Boolean algebra B and for every map f: } V \rightarrow B, \; \tilde{f}(a) \leq \tilde{f}(b),$$

and $\quad a \equiv b \;:\Leftrightarrow\; a \vdash b$ and $b \vdash a$.

Then $\equiv$ is an equivalence relation on L.

Let $\pi: L \rightarrow L/{\equiv}$ be the associated quotient map, and define

$$B(V) \;:=\; L/{\equiv}.$$

It is easy to see that there exists a binary relation $\leq'$ on B(V) such that

$$\text{for all } a, b \in B(V) : \pi(a) \leq' \pi(b) \text{ (only) if } a \vdash b.$$

Likewise, it is easy to see that there exist binary operations $\wedge'$ and $\vee'$ on B(V) such that for all a, b $\in$ B(V) : $\pi(a) \wedge' \pi(b) = \pi(a \wedge b)$

and for all a, b $\in$ B(V) : $\pi(a) \vee' \pi(b) = \pi(a \vee b)$,

and a unary operation $\neg'$ on B(V) such that

for all a, b $\in$ B(V) : $\neg'(\pi(a)) = \pi(\neg(a))$.

If we define

$\bot' := \pi(\bot_L)$

and $\top' := \pi(\top_L)$,

then it is also easy to see that (B(V), $\leq'$, $\wedge'$, $\vee'$, $\neg'$, $\bot'$, $\top'$) is a Boolean algebra. This Boolean algebra is called *the free Boolean algebra generated by (the elements of) V.*

It satisfies the following property :

A.10 Proposition (Using the names and notations of A.9)
For every Boolean algebra B and every map f: V $\to$ B there is one and only one Boolean translation f ' : B(V) $\to$ B such that

for all v $\in$ V, f(v) = f ' (π(v)).

(Proof: Elementary.)

A.11 Remark / Warning If any of the notations, symbols and terms used in A.9 and A.10 is used somewhere else in this thesis, we do typically *not* want to refer to A.9 and A.10. Exceptions to this rule are the symbol $\equiv$, the symbol $\vdash$, the notation B(V), and the term "the free Boolean algebra generated by (the elements of) V."

A.12 Proposition If V is finite, then B(V) is finite.
Proof: Let S denote the Boolean algebra $\mathcal{P}(\{0\})$.

By the definition of B(V), there are (at most) as many Boolean translations B(V) $\to$ S as there are maps V $\to$ S. If V is finite, there are only finitely many maps V $\to$ S. Hence, taking a closer look at the proof of A.8, we see that there exists an injective map B(V) $\to$ $\mathcal{P}$(X), for some finite set X. Hence, B(V) is finite.

□

Appendix B
Topology

This appendix contains preliminaries from elementary topology. Well-known results will be stated without proof as "facts". Proofs of these, as well as additional information about notions and definitions, can be found in any standard textbook on topology, for example in [Gaal 64].

B.1 Definition Let X be a set. A *topology on X* is a collection τ of subsets of X satisfying :

i) $X \in \tau$, $\varnothing \in \tau$,

ii) If a and $b \in \tau$, then $a \cap b \in \tau$,

iii) If $a_i \in \tau$ for all $i \in I$, then $\bigcup_{i \in I} a_i \in \tau$.

"Let (X, τ) be a topological space" means: let τ be a topology on X. Sometimes we just say "let X be a topological space". $a \subseteq X$ is called an *open* subset of X whenever $a \in \tau$.

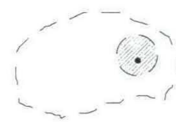
In this thesis, variable names like O, O', O_1, etc. are used exclusively for open sets. In addition, we will skip the word "open" as much as possible.

B.2 Definition A space X is called a *Hausdorff space* whenever for every p, $q \in X$ such that $q \neq p$ there exist O_p, $O_q \subseteq X$ such that

$$p \in O_p,\ q \in O_q \text{ and } O_p \cap O_q = \varnothing.$$

B.3 Example If X is a set, then $\mathcal{P}(X)$, the collection of all subsets of X, is a topology on X, called the *discrete* topology. The collection $\{\varnothing, X\}$ is also a topology on X, called the *trivial* topology.
Any set equipped with the discrete topology is a Hausdorff space. A set equipped with the trivial topology is not a Hausdorff space, unless the underlying set is empty or contains only one element.

B.4 Example The Euclidean plane, $\mathbb{R}^2$, is standardly assumed to be equipped with the *Euclidean topology*, defined as follows: $O \subseteq \mathbb{R}^2$ is open whenever for all $p \in O$ there is a circular-disc-without-boundary contained in O and containing p.



Hence, circular-discs-without-boundary are open, all unions of such sets are open and every open set is a union of such sets. That is, the open sets of the Euclidean plane are precisely the unions of circular-discs-without-boundary.

B.5 Example Likewise, $\mathbb{R}$ is canonically equipped with the topology consisting of all unions of *open intervals*.

Intuitively, a topology on a set X is a device that describes how the points of X are geometrically "glued together". Typically there exists more than one topology on every set, corresponding to different ways of glueing the points. Not all topological spaces are equally convincing as carriers of some geometrical intuition. For example, topologies on finite spaces are typically difficult to interpret geometrically. Such spaces are sometimes called "pathological spaces" (a notion that does not have a precise definition.)

B.6 Definition If (X, τ) is a topological space and $a \subseteq X$, then a inherits a topology from X, called the *induced topology* (*on* a) :

$$\tau_a := \{ O \cap a \mid O \in \tau \}$$

This amounts to $b \subseteq a$ being open in a (only) if there is some O, open in X, such that $b = O \cap a$.

(Again, it is easy to check that this defines a topology on a.)

For example, the interval $(1/2, 1]$ is not open in $\mathbb{R}$, but it is open in $[0, 1]$.

It is custom to generalize this definition, by calling any $b \subseteq X$ open in a whenever $b \cap a$ is open in a.

B.7 Definition Let (X, τ) be a topological space, and $a, b \subseteq X$.

a is *closed* (*in* X) whenever a^c is open (in X).

(a is *closed in* b whenever $a \cap b$ is closed in b with induced topology.)

a is *dense* (*in* X) whenever $O \neq \varnothing$ implies $O \cap a \neq \varnothing$.

(a is *dense in* b whenever $a \cap b$ is dense in b with induced topology.)

a is *nowhere dense* (*in* X) if there is no nonempty O in which a is dense.

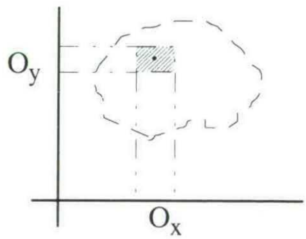
The *closure* of a, written as $\bar{a}$, is the smallest closed set containing a.

The *interior* of a, written a°, is the largest open set contained in a.

B.8 Facts a is dense in X (only) if $\bar{a} = X$.
If a is both dense and closed in X, then $a = X$.
For all $a \subseteq X$, $a^\circ \subseteq a \subseteq \bar{a}$.
If $a \subseteq b$, then $\bar{a} \subseteq \bar{b}$ and $a^\circ \subseteq b^\circ$.
If $a^\circ \subseteq b$, then $a^\circ \subseteq b^\circ$.

B.9 Definition An *isolated point* of a space X is a $p \in X$ such that $\{p\}$ is an open subset of X.
(For example, in IN with discrete topology, 0 is an isolated point. In IR, however, 0 is not an isolated point.)

B.10 Definition Let (X, τ_X) and (Y, τ_Y) be topological spaces. The *product topology* on $X \times Y$ is the smallest topology on this set containing $\tau_X \times \tau_Y := \{ O_X \times O_Y \mid O_X \in \tau_X \text{ and } O_Y \in \tau_Y \}$. Note that this set itself is not a topology on $X \times Y$, as it does not satisfy B.1 iii).
This amounts to $a \subseteq X \times Y$ being open (only) if for every $(x, y) \in a$ there are O_x and O_y, open in X and Y, respectively, such that $(x, y) \in O_x \times O_y$ and $O_x \times O_y \subseteq a$.



B.11 Facts The product topology on $IR \times IR$ is the same as the Euclidean topology. For every $c \in IR$, the induced topology on $\{c\} \times IR$ is the same as the standard topology on IR (see Example B.5).

B.12 Facts If X and Y are topological spaces, $a \subseteq X$ and $b \subseteq Y$, then

$$\overline{a \times b} = \bar{a} \times \bar{b}$$

(where $\overline{a \times b}$ denotes the closure *in X×Y* of $a \times b$, $\bar{a}$ denotes the closure *in X* of a, and $\bar{b}$ denotes the closure *in Y* of b).
Likewise, $(a \times b)^\circ = a^\circ \times b^\circ$.

B.13 Definition Let X be a topological space, and ~ an equivalence relation on X. Let π: X $\to$ X/~ be the quotient map. Then X/~ is standardly equipped with the *quotient topology*, defined as follows : $a \subseteq$ X/~ is open whenever $\pi^{-1}(a)$ is open in X.

B.14 Definition Let X and Y be topological spaces. A map f: X $\to$ Y is called *continuous* whenever for every O, open in Y, $f^{-1}(O)$ is open in X. It is called *open* whenever for every O, open in X, f(O) is open in Y.

B.15 Fact If X is a topological space and π: X $\to$ X/~ is a quotient map, then the quotient topology is the largest topology on X/~ for which π is continuous.

Open quotient maps

For Chapter 4, (14.1 and 14.2) we will need the following standard definitions and facts (in particular, Corollary B.20 and Proposition B.21.)

B.16 Definition For i = 1, 2, let X_i and Y_i be topological spaces and let f_i be a map $X_i \to Y_i$. Then the map $f_1 \times f_2$ is the map $X_1 \times X_2 \to Y_1 \times Y_2$ defined by

$$(f_1 \times f_2)((x_1, x_2)) = (f_1(x_1), f_2(x_2)) \quad \text{for all } (x_1, x_2) \in X_1 \times X_2.$$

This map is called the *product map* (of f_1 and f_2).

B.17 Facts The product map of two surjective maps is a surjective map. The product map of two continuous maps is continuous, the product map of two open maps is open.

B.18 Definition A map f: X $\to$ Y is called a *quotient map* whenever

i) f is surjective,

ii) for all $a \subseteq$ Y, O is open in Y (only) if $f^{-1}(O)$ is open in X.

(cf. Definition B.13.)

B.19 Fact If a map is surjective, continuous and open, then it is a quotient map.

B.20 Corollary The product of two open quotient maps is an open quotient map. (Proof: Every open quotient map is surjective, continuous and open. Now apply B.17 and B.19)

The topology of the set of Euclidean lines.

The collection G of lines of the Euclidean plane is the quotient space of a subset of $\mathbb{R}^3$, namely in the following way. We may identify the Euclidean plane with $\mathbb{R}^2$. Then every line ℓ in $\mathbb{R}^2$ is determined by an equation

$$px + qy + r = 0,$$

for some p, q, r $\in \mathbb{R}$.

(That is, there are p, q, r $\in \mathbb{R}$ such that $\ell = \{ (x,y) \mid px + qy + r = 0 \}$.

On the other hand, for every $(p, q, r) \in \mathbb{R}^3$ such that

$$(p, q) \neq (0, 0)$$

this equation determines a line in $\mathbb{R}^2$.

Moreover, the equations $p_1x + q_1y + r_1 = 0$ and $p_2x + q_2y + r_2 = 0$ determine the same line (only) if there is a $\lambda \in \mathbb{R}$ such that $\lambda \neq 0$ and

$$(\lambda p_1, \lambda q_1, \lambda r_1) = (p_2, q_2, r_2).$$

Define

$$H := \{ (p, q, r) \in \mathbb{R}^3 \mid (p, q) = (0, 0) \},$$

and, for (p_1, q_1, r_1) and $(p_2, q_2, r_2) \in \mathbb{R}^3 \setminus H$,

$$(p_1, q_1, r_1) \sim (p_2, q_2, r_2) \quad :\Leftrightarrow$$

there is a $\lambda \in \mathbb{R}$ such that $\lambda \neq 0$ and $(\lambda p_1, \lambda q_1, \lambda r_1) = (p_2, q_2, r_2)$.

Then there is a 1-to-1-correspondence between the elements of G and the elements of $(\mathbb{R}^3 \setminus H) / \sim$.

This latter set, in its turn, is standardly equipped with a topology (namely a quotient topology of an induced topology of the standard product topology on $\mathbb{R}^3$). By using the 1-to-1-correspondence, G is equipped with a topology.

The geometrical idea behind the above construction is the following.

Given a plane, E, choose a point P (in Euclidean space) not in E. Let O be open (in the Euclidean space) not containing P. For every point Q in O there is one and only one plane through P perpendicular to the line PQ. The intersection of this plane with E is either empty or a line. Thus, with every O is associated a set of lines in E (by ignoring that empty set). Now, the open sets of G are precisely the sets that arise in the way described above.

B.21 Proposition $\pi: \mathbb{R}^3 \setminus H \to G$ is an open quotient map.

Proof: If O is an open subset of $\mathbb{R}^3$, and $\lambda \in \mathbb{R}\setminus\{0\}$, then

$$\lambda O \; (:= \{ \lambda x \mid x \in O \})$$

is open in $\mathbb{R}^3$ as well.

Hence, suppose that O is an open subset of $\mathbb{R}^3 \setminus H$. Then

$$\pi^{-1}(\pi(O)) = \bigcup_{\lambda \in \mathbb{R}} \lambda O,$$

which is open in $\mathbb{R}^3$, by B1 iii).
Moreover, since H is a linear subspace of $\mathbb{R}^3$, $\pi^{-1}(\pi(O)) \subseteq \mathbb{R}^3 \setminus H$.
Hence, $\pi^{-1}(\pi(O))$ is open in $\mathbb{R}^3 \setminus H$. That is, $\pi(O)$ is open in $(\mathbb{R}^3 \setminus H) / \sim$.
Hence, π is an open map.

$\square$

Bibliography

[Abbott 69] J.C. Abbott, *Sets, Lattices and Boolean Algebras*, Allyn and Bacon, Boston, 1969.

[Adams 74] E.W. Adams, *The Logic of "Almost all"*, Journal of Philosophical Logic 3 (1974), 3-17.

[Adams 75] E.W. Adams, *The Logic of Conditionals*, Reidel, The Netherlands, 1975.

[AGM 85] C. E. Alchourrón, P. Gärdenfors. D. Makinson, *On the logic of theory change: partial meet contraction and revision functions*, The Journal of Symbolic Logic, 50 (2) 510-530, June 1985.

[Asher 95] N. Asher, *Commonsense Entailment: A Conditional Logic for Some Generics*, in: [Crocco *et al.* 95], p. 103-145.

[Asher & Morreau 91] N. Asher, M. Morreau, *Commonsense Entailment: a modal theory of nonmonotonic reasoning*, in: J. Mylopoulos, R. Reiter eds., Proceedings of the 12th Intl. Joint Conference on AI, Morgan Kauffman, California, 1991, p. 387-392.

[Aumann, Hart & Perry 96] R.J. Aumann, S. Hart, M. Perry, *The Absent-Minded Driver*, in: Proceedings of the 6th TARK, ed. Y. Shoham, Morgan Kaufmann, 1996, p. 97-116.

[Avron 92] A. Avron, *Whither Relevance Logic?*, Journal of Philosophical Logic 21 (1992) 243-281.

[Bacchus 90] F. Bacchus, *Representing and Reasoning with Probabilistic Knowledge: a logical approach to probabilities*, MIT Press, Cambridge, 1990.

[Bell 90] J. Bell, *Why the Frame Problem is not a Problem*, AICOM vol.3, no.1, March 1990.

[Bell 92] J. Bell, *Pragmatic Reasoning, a model-based theory*, in: Proceedings of the Applied Logic Conference "Logic at Work", Amsterdam 1992.

[Bell 95] J.L. Bell, *Infinitesimals and the Continuum*, The Mathematical Intelligencer, vol. 17 no. 2, 1995, p. 55-57.

[Benferhat *et al.* 93] S. Benferhat, C. Cayrol, D. Dubois, J. Lang, H. Prade, *Inconsistency management and prioritized syntax-based entailment*, in: Proceedings of the 13th IJCAI, 1993, p.640-645.

[Benthem 84] J. van Benthem, *Partiality and Nonmonotonicity in Classical Logic*, Stanford, CSLI, 1984.

[Bochenski 51] I.M. Bochenski, Ancient Formal Logic, North-Holland, Amsterdam, 1951, reprinted 1963.

[Boutilier 90] C. Boutilier, *Conditional Logics of Normality as Modal Systems*, in: Proceedings AAAI-90, AAAI Press / MIT Press, 1990, Vol. 2, p. 594-599.

[Boutilier 91] C. Boutilier, *Inaccessible worlds and Irrelevance: Preliminary Report*, in: Proceedings of the 12th IJCAI, 1991, p. 413-418.

[Boutilier 93] C. Boutilier, *Revision Sequences and Nested Conditionals*, in: Proceedings of the 13th IJCAI, 1993, p. 519-525.

[Boutilier 94a] C. Boutilier, *Unifying Default Reasoning and Belief Revision in A Modal Framework*, Artificial Intelligence 68 (1) (1994) 33-85.

[Boutilier 94b] C. Boutilier, *Conditional Logics of Normality: A Modal Approach*, Artificial Intelligence 68 (1) (1994) 87-154.

[Brewka 91] G. Brewka, *Nonmonotonic Reasoning: Logical Foundations of Commonsense*, Cambridge University Press, 1991.

[Crocco *et al.* 95] G. Crocco, L. Fariñas del Cerro, A. Herzig, *Conditionals: from Philosophy to Computer Science*, Clarendon Press, Oxford, 1995.

[Delgrande 87] J.P. Delgrande, *An approach to Default Reasoning Based on a First-Order Conditional Logic*, in: Proceedings AAAI-87, Seattle, WA, 1987, Vol. 1, p. 340-345.

[Dempster 67] A. Dempster, *Upper and Lower Probabilities induced by a multivalued mapping*, Annals of Mathematical Statistics, 38 (1967) 325-339.

[Dempster 68] A. Dempster, *A Generalization of Bayesian Inference*, Journal of the Royal Statistical Society, series B, 30 (1968) 295-247.

[Dujet & Vincent 95] Ch. Dujet, N. Vincent, *Force Implication: A new approach to human reasoning*, Fuzzy Sets and Systems 69 (1995) 53-63.

[Dung 95] P. M. Dung, *On the acceptability of arguments and its fundamental role in nonmonotonic reasoning, logic programming and n-person games*, Artificial Intelligence 77 (1995) 321-357.

[Eiter & Gottlob 93] T. Eiter, G. Gottlob, *The complexity of Nested Counterfactuals and Iterated Knowledge Base Revisions*, in: Proceedings of the 13th IJCAI, 1993, p. 526-531.

[Fine 85] K. Fine, *Arbitrary Objects*, Blackwell, Oxford, 1985.

[Flach 95] P.A. Flach, *Conjectures : an inquiry concerning the logic of induction*, Ph.D. thesis, Institute for Language Technology & Artificial Intelligence, Tilburg University, 1995.

[Freund & Lehmann 94] M. Freund, D. Lehmann, *Belief revision and rational inference*, report of the Leibniz Center for Research in Computer Science, July 1994.

[Gaal 64] S.A. Gaal, *Point Set Topology*, Academic Press, New York-London, 1964.

[Gabbay 95] D.M. Gabbay, *Conditional Implications and Non-monotonic Consequence*, in: [Crocco *et al.* 95], p. 337-359.

[Gärdenfors 87] P. Gärdenfors, *Variations on the Ramsey test: more triviality results*, Studia Logica 46 (1987) 321-327.

[Gärdenfors 92] P. Gärdenfors (ed.), *Belief Revision*, Cambridge Tracts in Theoretical Computer Science, Cambidge University Press, 1992.

[Gärdenfors & Makinson 94] P. Gärdenfors, D. Makinson, *Nonmonotonic inference based on expectations*, Artificial Intelligence 65 (1994) 197-245.

[Geffner 92] H. Geffner, *Default Reasoning: causal and conditional theories*, MIT Press, 1992.

[GMP 90] M. Goldszmidt, P. Morris, J. Pearl, *A Maximum Entropy Approach to Nonmonotonic Reasoning*, in: Proceedings of the 8th. AAAI, 1990, Vol. 2, p. 646-652.

[GMP 93] M. Goldszmidt, P. Morris, J. Pearl, A Maximum Entropy Approach to Nonmonotonic Reasoning, IEEE Transactions on Pattern Analysis and Machine Intelligence, Vol. 15, no. 3, March 1993, p. 220-232.

[Hacking 75] I. Hacking, *The Emergence of Probability: a Philosophical study of early ideas about Probability, Induction and Statistical Inference*, Cambridge University Press, London, 1975.

[Hendrickx 94] R.L.L. Hendrickx, *Over wiskunde en werkelijkheid*, Proefschrift Katholieke Universiteit Brabant, Tilburg, 1994.

[Hjelmslev 23] J. Hjelmslev, *Die Natürliche Geometrie. Vier Vorträge*, Abh. Math. Sem. Hamburg (2) 1923, p. 1-36.

[Indurkhya 92] B. Indurkhya, *Metaphor and Cognition*, Kluwer, 1992.

[Johnson-Laird 83] P.N. Johnson-Laird, *Mental Models*, Cambridge University Press, 1983.

[Jurjus 95] H.H. Jurjus, *On Nesting of a Nonmonotonic Conditional*, Computer Science Report 95/29, Eindhoven University of Technology, the Netherlands, 1995.

[Jurjus 97] H.H. Jurjus, *On Nesting of a Nonmonotonic Conditional*, in: Proceedings of the DGNMR97, E. Weydert, G. Brewka, C. Witteveen (eds.), MPI, Saarbrücken, 1997.

[Kandel & Lee 79] A. Kandel, S.C. Lee, *Fuzzy Switching and Automata: theory and applications*, New York: Crane Russak / London: Edward Arnold, 1979.

[Kandel 80] A. Kandel, *Fuzzy Dynamical Systems and the Nature of their Solutions*, in: Fuzzy Sets: Theory and Applications to Policy Analysis and Information Systems, eds. P.P. Wang, S.K. Chang, Plenum Press, New York / London, 1980, p. 93-121.

[Katsuno & Mendelzon 92] H. Katsuno, A.O. Mendelzon, *On the Difference between Updating a Knowledge Base and Revising it*, in: [Gärdenfors 92], p. 183-203.

[Katsuno & Satoh 95] H. Katsuno, *A Unified View of Consequence Relation, Belief Revision and Conditional Logic*, in: [Crocco *et al.* 95], p. 33-65.

[Kelley 55] J.L. Kelley, *General Topology*, Van Nostrand, 1955.

[KLM 90] S. Kraus, D. Lehmann, M. Magidor, *Nonmonotonic Reasoning, Preferential Models and Cumulative Logics*, Artificial Intelligence 44 (1990) 167-207.

[Konolige 87] K. Konolige, *On the Relation between Default and Autoepistemic Logic*, Stanford, CSLI, 1987.

[Lamarre 91] Ph. Lamarre, *S4 as the Conditional Logic of Nonmonotonicity*, in: Proceedings of the 2nd International Conference (KR91), J. Allen, R. Fikes, E. Sandewall (eds.), Kaufmann Publishers, 1991, p. 357-367.

[Lehmann 92] D. Lehmann, *Plausibility Logic*, Report of the Leibniz Center for Research in Computer Science, February 1992.

[Lehmann 95] D. Lehmann, *Belief revision, revised*, in: C.S. Mellish (ed.), Proceedings of the 14th IJCAI, Morgan Kaufmann Publishers, Montréal, 1995, Vol. 2, p. 1534-1540.

[Lewis 73] D. Lewis, *Counterfactuals*, Blackwell, Oxford, 1973.

[LM 92] D. Lehmann, M. Magidor, *What does a conditional knowledge base entail?*, Artificial Intelligence 55 (1992) 1-60.

[Makinson 90] D. Makinson, *The Gärdenfors Impossibility Theorem in non-monotonic contexts*, Studia Logica 49 (1990) 1-6.

[Makinson 93] D. Makinson, *Five Faces of Minimality*, Studia Logica 52 (1993) 339-379.

[Makinson 94] D. Makinson, *General Patterns in Nonmonotonic Reasoning*, in: D.M. Gabbay et al. (eds.), Handbook of Logic in Artificial Intelligence and Logic Programming, Clarendon Press, Oxford 1994, Vol.3, p. 35-110.

[McCarthy 80] J. McCarthy, *Circumscription - a form of non-monotonic reasoning*, Artificial Intelligence 13 (1980) 27-39.

[T. McCarthy 94] T. McCarthy, *Self-Reference and Incompleteness in a Non-Monotonic Setting*, Journal of Philosophical Logic 23 (1994) 423-449.

[McGee 94] V. McGee, *Learning the Impossible*, in: E. Eells, B. Skyrms (eds.) Probability and Conditionals, Cambridge University Press, 1994, p. 179-199.

[Moore 85] R.C. Moore, *Semantical considerations on nonmonotonic logic*, Artificial Intelligence 25 (1985) 75-94.

[Moore 88] R.C. Moore, *Autoepistemic Logic*, in: Ph. Smets (ed.), Non-standard Logics for Automated Reasoning, London, Academic Press, 1988, p. 105-136.

[Morreau 91] see [Asher & Morreau 91]

[Morreau 92] M. Morreau, *Epistemic Semantics for Counterfactuals*, Journal of Philosophical Logic 21 (1992) 33-62.

[Morreau 95] M. Morreau, *Fainthearted Conditionals*, in: Proceedings of the Second Dutch-German Workshop on Non-monotonic Reasoning, p. 125-134.

[Nait Abdallah 95] A. Nait Abdallah, *The Logic of Partial Information*, Berlin / Heidelberg / New York, Springer, 1995.

[Neumann 55] J. von Neumann, *Probabilistic Logics and the Synthesis of Reliable Organisms from Unreliable Components*, in: C.E. Shannon, J. McCarthy (eds.), Automata Studies, Princeton University Press, 1956, 2nd print 1958, p. 43-98.

[Nieuwint 90] P.J.G.M. Nieuwint, *Implication in Natural Language and in Logic*, Methodology and Science 23 (1990) 200-216.

[Nieuwint 92] P.J.G.M. Nieuwint, *On conditionals: a study of conditional sentences in English and Dutch*, proefschrift Katholieke Universiteit Brabant, Tilburg, 1992.

[Novak 92] V. Novák, *The Alternative Mathematical Model of Linguistic Semantics and Pragmatics*, IFSR International Series on Systems Science & Engineering, vol. 8, Plenum Press, NY/London 1992, Chapter 2: The Alternative Set Theory (p. 15-48).

[Parikh 71] R. Parikh, *Existence and Feasibility in Arithmetic*, Journal of Symbolic Logic 36 (3) 494-508, 1971.

[Pinkas 95] G. Pinkas, *Reasoning, nonmonotonicity and learning in connectionist networks that capture propositional knowledge*, Artificial Intelligence 77 (1995) 203-247.

[Pitts 82] A.M. Pitts, *Fuzzy Sets do not form a Topos*, Fuzzy Sets and Systems 8 (1982) 101-104.

[Pollock 94] J.L. Pollock, *Justification and Defeat*, Artificial Intelligence 67 (1994) 377-407.

[Poole 88] D. Poole, *A logical framework for default reasoning*, Artificial Intelligence 36 (1988) 27-47.

[Prakken 95] H. Prakken, *Argumentation with defeasible conditionals: a preliminary report*, in: Proceedings of the Second Dutch-German workshop on Non-monotonic Reasoning, Utrecht, 1995, p. 160-167.

[Priest 80] G. Priest, *Sense, Entailment and Modus Ponens*, Journal of Philosophical Logic 9 (1980) 415-435.

[Rauszer 92] C. Rauszer, Rough set theory, lecture, Tilburg, December 1992.

[Robert 88] A. Robert, *Nonstandard Analysis*, John Wiley and Sons, 1988.

[Ryan & Schobbens 96] M. Ryan, P.-Y. Schobbens, *Counterfactuals and updates as inverse modalities*, in: Proceedings of the 6th TARK, 1996, p. 163-173.

[Sainsbury 88] R.M. Sainsbury, *Paradoxes*, Cambridge University Press, 1988.

[Servi 92] G.F. Servi, *Nonmonotonic Consequence Based on Intuitionistic Logic*, Journal of Symbolic Logic, 57 (4) 1176-1197, December 1992.

[Shafer 76] G. Shafer, *A Mathematical Theory of Evidence*, Princeton University Press, 1976.

[Shoham 88] Y. Shoham, *Efficient Reasoning about Rich Temporary Domains*, Journal of Philosophical Logic 17 (1988) 443-474.

[de Swart & Nederpelt 92] H.C.M. de Swart, R.P. Nederpelt, *Implication - A Survey of the Different Logical Analyses of "if ..., then ..."*, in: Nieuw Archief voor Wiskunde, 1992, p.77-104.

[Veldman 91] W.H.M. Veldman, *Intuïtionistische wiskunde*, syllabus van het Mathematisch Instituut, Nijmegen, 1991.

[Weydert 93] E. Weydert, *Default Quantifiers*, in: Working Notes from the Dutch/German Workshop on Nonmonotonic Reasoning Techniques and their Applications, RWTH Aachen, 1993.

[Weydert 94] E. Weydert, *Hyperrational Conditionals - Monotonic Reasoning about Nested Default Conditionals*, in: G. Lakemeyer, B. Nebel (eds.), Foundations of Knowledge Representation and Reasoning, Springer Verlag, 1994.

[Yessenin-Volpin 70] A.S. Yessenin-Volpin, *The Ultraintuitionistic Criticism and the Antitraditional Program for Foundations of Mathematics*, in: Intuitionism and Proof-Theory, Proceedings of the Conference at Buffalo 1968, North-Holland, Amsterdam 1970.

[Zadeh 65] L.A. Zadeh, *Fuzzy Sets*, Information and Control 8 (1965) 338-353.

[Zadeh 75] L.A. Zadeh, *Fuzzy Logic and Approximate Reasoning*, Synthese 30 (1975) 407-428.

[Zadeh 88a] L.A. Zadeh, *Fuzzy Logic*, Stanford, CSLI, 1988.

[Zadeh 88b] L.A. Zadeh, *Dispositional Logic and Commonsense Reasoning*, Stanford, CSLI, 1988.

Samenvatting

Niet-monotone logica bestudeert redeneervormen waarbij eenmaal getrokken conclusies kunnen worden teruggetrokken op grond van nieuwe informatie. Menselijk redeneren van alledag is niet-monotoon. Wiskundig redeneren daarentegen is monotoon, in die zin dat daar uit méér gegevens nooit mínder conclusies getrokken kunnen worden. Hoewel het niet moeilijk is om met wiskundige hulpmiddelen kunstmatige niet-monotone formalismen te ontwerpen, blijkt menselijk redeneergedrag keer op keer intelligenter: doelmatiger en beter aangepast aan alledaagse problemen. Weliswaar voldoet ook menselijk niet-monotoon redeneren aan zekere algemene wetmatigheden, maar deze wetmatigheden lijken niet voor precieze wiskundige beschrijving vatbaar. In de wiskunde is een regel ofwel geldig, en dan heeft hij geen uitzonderingen, of hij heeft uitzonderingen, maar dan is hij niet geldig. In het dagelijks leven hanteren we daarentegen voortdurend regels die uitzonderingen hebben, maar "in-de-regel" geldig zijn. Ook de wetmatigheden van (menselijk) niet-monotoon redeneren lijken het karakter te hebben van regels met mogelijke uitzonderingen.

In dit proefschrift worden een aantal manieren beschreven om de uitdrukkingskracht van de wiskundige taal te vergroten, zodat we formele axioma's, die we willen opvatten als regels-met-mogelijke-uitzonderingen, toch van een precieze betekenis kunnen voorzien. Gemeenschappelijke kern van de systemen is het topologische begrip "volle deelverzameling", dat geïnspireerd is op intuïties uit de Euclidische meetkunde (verrassend genoeg een traditioneel wiskundig onderwerp, en van oudsher een belangrijke inspiratiebron voor klassieke, "monotone" logica). Met dit begrip is het eenvoudig om betekenis te geven aan "verbiddellijke" of "genaakbare" implicatie (Eng. *defeasible* implication; notatie "$\rightarrow$" of "$\mid\!\sim$").

We vinden een volledig stelsel van axioma's, waarbij we voor het bewijs gebruik maken van het werk van [KLM 90]. Het blijkt overigens om het zelfde axiomastelsel te gaan. Dit topologische begrip wordt vervolgens (hoofdstuk 3) gebruikt om ook betekenis te geven aan "geneste" implicaties, zoals bijv. in $(a \rightarrow b \wedge c) \rightarrow (a \rightarrow b)$ of $((a \rightarrow b) \wedge a) \rightarrow b$. We kunnen het resultaat op twee manieren bekijken. Op de eerste plaats kunnen we het formalisme beschouwen als een wiskundige karikatuur van een persoon die sommige inferentie regels beschouwt als geldig-op-mogelijke-uitzonderingen-na. In ons formalisme volgt de persoon alle regels van klassieke propositie logica, inclusief modus ponens en de monotonie regel $(a \rightarrow b) \rightarrow (a \wedge c \rightarrow b)$. Gezien door de ogen van een wiskundig redenerend buitenstaander, die de monotonie regel als regel-zonder-uitzonderingen interpreteert, redeneert de persoon echter niet-monotoon; de monotonie regel is niet geldig-zonder-uitzonderingen. Wat uit ons formalisme blijkt is, dat het

mogelijk is een ander implicatie-begrip te hanteren dan het "klassieke", met behoud van alle klassieke logische wetten, mits we zo consequent zijn het alternatieve implicatie-begrip ook te gebruiken om die logische wetten te interpreteren.
In de tweede plaats kunnen we het formalisme beschouwen als een manier om "genaakbare" implicatie als logisch voegteken toe te voegen aan klassieke propositie logica. Als gevolg van de betekenis van genaakbare implicatie zal de inferentie relatie dan niet-monotoon zijn: uit "a $\rightarrow$ b en a" is "b" afleidbaar, maar uit "a $\rightarrow$ b, a en niet-b" is "b" niet afleidbaar. Zo bekeken, hangt de niet-monotonie van deze inferentie relatie niet samen met "niet-deductief redeneren" of met "voorbarige conclusies". Het is een consequentie van de betekenis van het voegteken "$\rightarrow$". De inferentie relatie beschrijft een vorm van correct, deductief redeneren die niettemin niet-monotoon is.

De aanpak van hoofdstuk 3, die genaakbare inferentieregels "invoert" via geneste implicaties, heeft echter twee nadelen. Allereerst zou het formalisme duidelijker zijn, wanneer we genaakbare inferentie-regels konden formuleren als gekwantificeerde uitspraken over proposities. (De aard van het formalisme is van belang, omdat bovenstaande conclusies van hoofdstuk 3 het gevolg zouden kunnen zijn van onrealistische details in de formalisering. Zo bleek in §12, dat sommige varianten van het formalisme, in het bijzonder O_π-validity, niet tot dezelfde conclusies leiden.)
In de tweede plaats is het gedrag van het systeem van hoofdstuk 3 niet altijd in overeenstemming met wat we zouden verwachten (zie Example 11.11). Formalismen die zich op dit punt beter gedragen kunnen echter niet op hetzelfde principe gebaseerd zijn (d.w.z. een implicatie-operator en geneste implicatie).

Gelukkig kan het topologische begrip "volle deelverzameling" ook gebruikt worden om universele kwantificatie met mogelijke uitzonderingen te definiëren. Dit wordt uitgewerkt in hoofdstuk 4, onder andere door een aantal voorbeelden (§15), die laten zien dat dit onderwerp op zichzelf al van belang is voor het bestuderen van practische argumentatie. Deze niet-standaard interpretatie van universele kwantificatie gebruiken we vervolgens in §16, waar we genaakbare inferentie-regels op directe wijze formuleren als gekwantificeerde uitspraken over het zinnen-repertoire van een redenerend persoon. We passen het principe toe op de regels van klassieke propositie logica, en laten zien dat het resultaat de conclusies van hoofdstuk 3 bevestigt: de regels van klassieke propositie logica kunnen niet gezien worden als definitie of "karakterisering" van de gebruikelijke interpretatie van inferentie en deduceerbaarheid, omdat de interpretatie van de regels het

juiste begrip van deze noties vooronderstelt. We moeten al weten wat implicatie-zonder-uitzonderingen betekent, voor we de regels op de bedoelde wijze kunnen interpreteren.
De aanpak van §16 is bovendien bruikbaar om, naast klassieke propositie logica, ook allerlei andere axioma-stelsels op gelijke wijze te behandelen. Aan het onderzoeken van andere axioma-stelsels komen we echter in dit proefschrift niet toe (waarmee meteen een mogelijke richting voor verder onderzoek gegeven is).

Wie nader kennis wil nemen van de inhoud van het proefschrift, zonder het boekje helemaal te lezen, wordt aangeraden de eerste paragrafen van elk hoofdstuk te bestuderen, te weten, §1, §5, §9 (+ §10), §14 en §16.

Curriculum Vitae

After secondary school (Stedelijk Gymnasium in 's-Hertogenbosch), the author studied mathematics at the Catholic University of Nijmegen and piano at 'Brabants Conservatorium' in Tilburg. He graduated in mathematics in 1992, cum laude. Also in 1992, he was appointed as a Research Trainee at the Department of Philosophy of Tilburg University, working on a project called 'plausible reasoning' that lead to this thesis. Herman Jurjus has lived most of his life in 's-Hertogenbosch, but has recently moved to the city of Ravenstein.

Non-monotonic logic originally arose as a subdiscipline of artificial intelligence. In *The Exception Proves the Rule* the phenomenon of non-monotory is studied from a mathematical point of view.

Among other things, it is asked what would happen if we interpret laws of logic not as mathematical axioms, but as rules - with - possible - exceptions.

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